

Context-Content Systems of Random Variables: The Contextuality-by-Default Theory

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This paper provides a systematic yet accessible presentation of the Contextuality-by-Default theory. The consideration is confined to finite systems of categorical random variables, which allows us to focus on the basics of the theory without using full-scale measure-theoretic language. Contextuality-by-Default is a theory of random variables identified by their contents and their contexts, so that two variables have a joint distribution if and only if they share a context. Intuitively, the content of a random variable is the entity the random variable measures or responds to, while the context is formed by the conditions under which these measurements or responses are obtained. A system of random variables consists of stochastically unrelated “bunches,” each of which is a set of jointly distributed random variables sharing a context. The variables that have the same content in different contexts form “connections” between the bunches. A probabilistic coupling of this system is a set of random variables obtained by imposing a joint distribution on the stochastically unrelated bunches. A system is considered noncontextual or contextual according to whether it can or cannot be coupled so that the joint distributions imposed on its connections possess a certain property (in the present version of the theory, “maximality”). We present a criterion of contextuality for a special class of systems of random variables, called cyclic systems. We also introduce a general measure of contextuality that makes use of (quasi-)couplings whose distributions may involve negative numbers or numbers greater than 1 in place of probabilities.

KEYWORDS: contextuality, couplings, connectedness, random variables.

1. INTRODUCTION

Contextuality-by-Default (CbD) is an approach to probability theory, specifically, to the theory of random variables. CbD is not a model of empirical phenomena, and it cannot be corroborated or falsified by empirical data. However, it provides a sophisticated conceptual framework in which one can describe empirical data and formulate models that involve random variables.

In Kolmogorovian Probability Theory (KPT) random variables are understood as measurable functions mapping from one (domain) probability space into another (codomain) probability space. CbD can be viewed as a theory within the framework of KPT if the latter is understood as allowing for multiple domain probability spaces, freely introducible and unrelated to each other. However, CbD can also be (in fact, is better) formulated with no reference to domain probability spaces, with random variables understood as entities identified by their probability distributions and their unique labels within what can be called sets of random variables “in existence” or “in play.”

Although one cannot deal with probability distributions without the full-fledged measure-theoretic language, we avoid technicalities some readers could find inhibitive by focusing in this paper on *finite* systems of *categorical random variables* (those with finite numbers of possible values). Virtually all of the content of this paper, however, is generalizable *mutatis mutandis* to arbitrary systems of arbitrary random entities.

1.1. A convention

In the following we introduce sets of random variables classified in two ways, by their *contexts* and by their *contents*, and we continue to speak of contexts and contents throughout the paper. The two terms combine nicely, but they are also easily confused in reading. For this reason, in this paper we do violence to English grammar and write “conteXt” and “conteNt” when we use these words as special terms.

1.2. Two conteNts in two conteXts

We begin with a simple example. A person randomly chosen from some population is asked two questions, q and q' . Say, q = “Do you like bees?” and q' = “Do you like to smell flowers?”. The answer to the first question (Yes or No) is a random variable whose *identity* (that which allows one to uniquely identify it within the class of all random variables being considered) clearly includes q , so it can be denoted R_q . We will refer to the question q as the *conteNt* of the random variable R_q . The second random variable then can be denoted $R_{q'}$, and its conteNt is q' . The set of all random variables being considered here consists of R_q and $R_{q'}$, and we do not confuse them because they have distinct conteNts: we know which of the two responses answers which question.

The two random variables have a joint distribution that can be presented, because they are binary, by values

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of the three probabilities

$$\Pr[R_q = \text{Yes}], \quad \Pr[R_{q'} = \text{Yes}],$$

$$\Pr[R_q = \text{Yes and } R_{q'} = \text{Yes}].$$

The joint distribution exists because the two responses, R_q and $R_{q'}$, occur together in a well-defined empirical sense: the empirical sense of “togetherness” of the responses here is “to be given by one and the same person.” In other situations the empirical meaning can be different, e.g., “to be recorded in the same trial.”

Our example is too simple for our purposes. Let us assume therefore that the two questions q, q' are asked under varying controlled conditions, e.g., one randomly chosen person can be asked these questions after having watched a movie about the killer bees spreading northwards (let us call this condition c), another after watching a movie about deciphering the waggle dances of the honey bees (c'). Most people would consider q as one and the same question whether posed under the condition c or the condition c' ; and the same applies to the question q' . In other words, the conteNts q and q' of the two respective random variables would normally be considered unchanged by the conditions c and c' .

However, the random variables themselves (the responses) are clearly affected by these conditions. In particular, nothing guarantees that the joint distribution of $(R_q, R_{q'})$ will be the same under the two conditions. It is necessary therefore to include c and c' in the description of the random variables representing the responses. We will call c and c' *conteXts* of (or for) the corresponding random variables and present them as $R_q^c, R_{q'}^c, R_q^{c'}, R_{q'}^{c'}$. There are now four random variables in play, and we do not confuse them because each of them is uniquely identified by its conteNt and its conteXt.

1.3. Jointly distributed versus stochastically unrelated random variables

In each of the two conteXts, the two random variables are jointly distributed, i.e., we have well-defined probabilities

$$\left. \begin{aligned} &\Pr[R_q^c = \text{Yes}], \\ &\Pr[R_{q'}^c = \text{Yes}], \\ &\Pr[R_q^c = \text{Yes and } R_{q'}^c = \text{Yes}] \end{aligned} \right\} \text{ in conteXt } c,$$

and

$$\left. \begin{aligned} &\Pr[R_q^{c'} = \text{Yes}], \\ &\Pr[R_{q'}^{c'} = \text{Yes}], \\ &\Pr[R_q^{c'} = \text{Yes and } R_{q'}^{c'} = \text{Yes}] \end{aligned} \right\} \text{ in conteXt } c'.$$

No joint probabilities, however, are defined between the random variables picked from different conteXts. We cannot determine such probabilities as

$$\Pr[R_q^c = \text{Yes and } R_{q'}^{c'} = \text{Yes}],$$

$$\Pr[R_q^c = \text{Yes and } R_{q'}^c = \text{Yes}],$$

$$\Pr[R_q^c = \text{Yes and } R_q^{c'} = \text{Yes and } R_{q'}^{c'} = \text{Yes}],$$

etc.

We express this important fact by saying that any two variables recorded in different conteXts are *stochastically unrelated*. The reason for stochastic unrelatedness is simple: no random variable in conteXt c can co-occur with any random variable in conteXt c' in the same empirical sense in which two responses co-occur within either of these conteXts, because c and c' are mutually exclusive conditions. The empirical sense of co-occurrence in our example is “to be given by the same person,” and we have assumed that a randomly chosen person is either shown one movie or another. If some respondents were allowed to watch both movies before responding, we would have to redefine the classification of our random variables by introducing a third conteXt, $c'' = (c, c')$. We would then have three pairwise mutually exclusive conteXts, c, c', c'' , and six random variables, $R_q^c, R_{q'}^c, R_q^{c'}, R_{q'}^{c'}, R_q^{c''}, R_{q'}^{c''}$, such that, e.g., $R_q^{c''}$ is jointly distributed with $R_{q'}^{c''}$ but not with R_q^c .

In case one is tempted to consider joint probabilities involving R_q^c and $R_{q'}^{c'}$ simply equal to zero (because these two responses never co-occur), this thought should be dismissed. Indeed, then all four joint probabilities,

$$\Pr[R_q^c = \text{Yes and } R_{q'}^{c'} = \text{Yes}],$$

$$\Pr[R_q^c = \text{Yes and } R_{q'}^{c'} = \text{No}],$$

$$\Pr[R_q^c = \text{No and } R_{q'}^{c'} = \text{Yes}],$$

$$\Pr[R_q^c = \text{No and } R_{q'}^{c'} = \text{No}],$$

would have to be equal to zero, which is not possible as they should sum to 1. These probabilities are not zero, they are *undefined*.

1.4. Bunches and connections in conteXt-conteNt matrices

The picture of the system consisting of our four random variables is now complete. Let us call this system \mathcal{A} . It is an example of a *conteXt-conteNt (c-c) system* of random variables, and it can be schematically presented in the

R_q^c	$R_{q'}^c$	c
$R_q^{c'}$	$R_{q'}^{c'}$	c'
q	q'	$\boxed{\mathcal{A}}$

Figure 1. A conteXt-conteNt matrix for system \mathcal{A} in our opening example. The system consists of two bunches $R^c = (R_q^c, R_{q'}^c)$, $R^{c'} = (R_q^{c'}, R_{q'}^{c'})$ defined by (or defining) the conteXts c and c' , respectively. The notation $R^c, R^{c'}$ reflects the fact that each bunch is a single random variable in its own right, because its components are jointly distributed. The system has two connections $(R_q^c, R_{q'}^c)$, $(R_q^{c'}, R_{q'}^{c'})$ defined by (or defining) the conteNts q and q' , respectively. The connections are not random variables because their components are stochastically unrelated. System \mathcal{A} may be contextual or noncontextual, depending on the distributions of the bunches $R^c, R^{c'}$.

form of the *conteXt-conteNt (c-c) matrix* in Fig. 1. All random variables in a c-c system are double-indexed: the lower index indicates their conteNt, the upper index indicates their conteXt. The random variables within each conteXt are jointly distributed, they form what we call a *bunch* (of random variables). One bunch corresponds to one conteXt and occupies one row of the c-c matrix. Any two variables that belong to different bunches are stochastically unrelated. However, a random variable in conteXt c may have a counterpart in conteXt c' that shares the same conteNt with it. In the system \mathcal{A} this is true for each of the two random variables in c (or c'): R_q^c and $R_{q'}^c$ represent answers to one and the same question, and so do $R_q^{c'}$ and $R_{q'}^{c'}$. The set (in our example, the pair) of all random variables sharing the same conteNt is called a *connection* (because they bridge stochastically unrelated bunches). One connection, in our example $(R_q^c, R_{q'}^c)$ or $(R_q^{c'}, R_{q'}^{c'})$, corresponds to one conteNt and occupies one column of the c-c matrix.

We will see in Section 5 that \mathcal{A} is the simplest system within the class of so-called cyclic systems. It can, of course, model more than the opening example with bees, flowers, and movies. The variety of possible applications is great, both within psychology and without. The conteNts q, q' can be two physical properties, e.g., spins of a particle measured at two moments in time separated by a fixed interval. The conteXts c, c' can then be, respectively, the presence and absence of a third measurement made prior to these two measurements. Alternatively, c and c' could be two orders in which the two measurements are conducted: c standing for “first q then q' ” and c' for “first q' then q .” In sociology and psychology the prominently studied analogue of the latter example is the paradigm in which two questions are posed in two possible orders (Moore, 2001; Wang & Busemeyer, 2013; Wang et al., 2014). One can also think of questions posed in two different forms, in two different languages, or asked

R_1^1	R_2^1	\cdot	c_1
R_1^2	R_2^2	R_3^2	c_2
R_1^3	\cdot	R_3^3	c_3
q_1	q_2	q_3	$\boxed{\mathcal{B}}$

Figure 2. A c-c matrix representation of a c-c system \mathcal{B} of random variables. The seven random variables are grouped into three bunches (shown by the rows of the matrix) and into three connections (shown by the columns). The bunches are defined by (or define) three conteXts. The connections are defined by (or define) three conteNts. The empty cells (shown with a dot for emphasis) correspond to the cases when a given conteNt is not represented (measured, responded to) in a given conteXt. The variables within a bunch are jointly distributed, so we have three random variables $R^1 = (R_1^1, R_2^1)$, $R^2 = (R_1^2, R_2^2, R_3^2)$, and $R^3 = (R_1^3, R_3^3)$. The connections (R_1^1, R_1^2, R_1^3) , (R_2^1, R_2^2) , and (R_3^2, R_3^3) are not random variables because no two random variables within a connection are jointly distributed.

of the representatives of two distinct populations (say, male and female). There is also an inexhaustible variety of psychophysical applications. For instance, q and q' may be visual stimuli, and c, c' may be any two variants of the conditions under which they are presented, such as the time interval or spatial separation between them, or two versions of a previously presented adapting stimulus.

Fig. 2 shows a c-c matrix representation of a more complex system, with three bunches and three connections.

A generalization to arbitrary c-c systems should be obvious: given a set of conteXts and a set of conteNts the cells in a c-c matrix can be filled (or left empty) in all possible ways, although constraints could be imposed to exclude matrices that are uninteresting for contextuality analysis (e.g., empty matrices, or those with a single random variable in a connection or in a bunch).

1.5. ConteXts and conteNts are non-unique but distinct from each other

How do we know that in our opening example the question q and not the movie c determines the conteNt of the response, viewed as a random variable? How do we know that the movie c and not the question q determines the conteXt of this response? The answer is: we don't. Some theory or tradition outside the mathematical theory of Cbd tells us what the conteXts and the conteNts in a given situation are, and then the mathematical computations may commence. In these computations, whatever conteXts and conteNts are given to us, they are treated as strictly distinct entities because the respective bunches and connections they define are fundamentally different: bunches are (multicomponent) random variables, while connections are groups of pairwise stochastically unre-

$R_{q_1}^c$	$R_{q_2}^c$	\cdot	\cdot	c
\cdot	\cdot	$R_{q_3}^{c'}$	$R_{q_4}^{c'}$	c'

$q_1 = (q, c) \quad q_2 = (q', c) \quad q_3 = (q, c') \quad q_4 = (q', c')$ \mathcal{A}'

Figure 3. The same opening example as in Fig. 1, but represented by a different system, \mathcal{A}' . In this system the conteXts are the same as in \mathcal{A} , but each new conteNt (question) includes as its part the original conteXt in which it occurs (the movie watched). The joint distributions within the two bunches remain unchanged, but the system loses connections between the bunches. Such a system is trivially noncontextual.

lated ones.

It would be a completely different system if the conteNts in our opening example were defined not just by the question asked but also by the movie previously watched. The c-c matrix would then be as shown in Fig. 3. No conteNt in the system \mathcal{A}' occurs more than once, so there is nothing to bridge the two bunches. It is not wrong to present the experiment with the questions and movies in this way, it may very well be the best way of treating this situation from the point of view of some empirical model, but the resulting system is not interesting for our contextuality analysis. The latter is yet to be introduced, but it should be sufficiently clear if we say that the system \mathcal{A}' is uninteresting because contextuality pertains to how the random variables that share conteNts differ in different conteXts.

To prevent turning this discussion into a game of superficial semantics, it would not do to point out that q_1 and q_3 in the system \mathcal{A}' share “part” of their conteNt, and hence $R_{q_1}^c$ and $R_{q_3}^{c'}$ can be related to each other on these grounds. If $R_{q_1}^c$ and $R_{q_3}^{c'}$ are members of the same connection, then they should have the same conteNt, by definition. A conteNts is, logically, merely a label for a connection.

A symmetrical opposite of the system \mathcal{A}' is to include the questions asked into the conteXts in which they are being asked. This creates the c-c matrix \mathcal{A}'' shown in Fig. 4. Since the empirical meaning of co-occurrence in our example is “to be given by the same person,” representing our opening example by the system \mathcal{A}'' amounts to simply ignoring the observed joint events. One only records (and estimates probabilities of) the individual events, as if the paired questions were asked separately of different respondents. This would not be a reasonable way of representing the situation (as it involves ignoring available information), but it is logically possible.

The system \mathcal{A}'' becomes a reasonable representation, however, in fact the only “natural” one, if the empirical procedure is modified and the questions are indeed asked one at a time rather than in pairs. Then the responses to questions about the bees and about the flowers, whether they are given after having watched the same movie or different movies, come from different respondents, and

$R_q^{c_1}$	\cdot	$c_1 = (c, q)$
$R_q^{c_2}$	\cdot	$c_2 = (c', q)$
\cdot	$R_{q'}^{c_3}$	$c_3 = (c, q')$
\cdot	$R_{q'}^{c_4}$	$c_4 = (c', q')$

$q \quad q'$ \mathcal{A}''

Figure 4. The same opening example as in Fig. 1, but represented by a different system, \mathcal{A}'' . In this system the conteNts are the same as in \mathcal{A} , but each new conteXt is identified by both the movie watched and the question asked. The connections are well-defined here, but each bunch contains a single variable. Such a system is trivially noncontextual. This system would more aptly be used to describe the experiment in which the questions are posed one question per respondent rather than in pairs.

$R_{q_1}^{c_1}$	\cdot	\cdot	\cdot	$c_1 = (c, q)$
\cdot	$R_{q_2}^{c_2}$	\cdot	\cdot	$c_2 = (c', q)$
\cdot	\cdot	$R_{q_3}^{c_3}$	\cdot	$c_3 = (c, q')$
\cdot	\cdot	\cdot	$R_{q_4}^{c_4}$	$c_4 = (c', q')$

$q_1 = (q, c) \quad q_2 = (q, c') \quad q_3 = (q', c) \quad q_4 = (q', c')$ \mathcal{A}'''

Figure 5. A combination of the modifications shown in Figs. 4 and 3: the original conteNts are treated as part of new conteXts, and the original conteXts are treated as part of new conteNts: the system consists of four unrelated to each other random variables, and it is trivially noncontextual. The order in which the q ’s and c ’s are listed in the definitions of the conteXts and the conteNts is immaterial.

their joint probabilities are undefined.

The system \mathcal{A}'' has the same connections as \mathcal{A} , but it is as uninteresting from the point of view of contextuality analysis as the system \mathcal{A}' . A system without joint distributions (i.e., one in which every bunch contains a single random variable) is always trivially noncontextual.

For completeness, we should also consider a radical point of view that combines those in the systems \mathcal{A}'' and \mathcal{A}' . It is shown in Fig. 5: every conteXt and every conteNt include information about both the question being asked and the conditions under which it is asked. This creates four unique conteNts in a one-to-one correspondence with four unique conteXts.

1.6. Multiple connections crossing multiple bunches

There is another aspect of the non-uniqueness of representing an empirical situation by a c-c system. As we see in both system \mathcal{A} of Fig. 1 and system \mathcal{B} of Fig. 2, two

R_1^1	R_2^1	\cdot	R_{12}^1	c_1
R_1^2	R_2^2	R_3^2	R_{12}^2	c_2
R_1^3	\cdot	R_3^3	\cdot	c_3
q_1	q_2	q_3	q_{12}	$\boxed{\mathcal{B}'}$

Figure 6. If one wishes to establish connection between the pair (R_1^1, R_2^1) and the pair (R_1^2, R_2^2) , the system \mathcal{B} of Fig. 2 has to be redefined: a new conteNt, q_{12} , should be created that adds $R_{12}^1 = (R_1^1, R_2^1)$ and $R_{12}^2 = (R_1^2, R_2^2)$ to the two respective bunches (for c_1 and c_2) as separate variables, each of which is jointly distributed with the other variables within the same bunch.

R_1^1	R_2^1	\cdot	R_{12}^1	\cdot	c_1
R_1^2	R_2^2	R_3^2	R_{12}^2	R_{13}^2	c_2
R_1^3	\cdot	R_3^3	\cdot	R_{13}^3	c_3
q_1	q_2	q_3	q_{12}	q_{13}	$\boxed{\mathcal{B}''}$

Figure 7. Further redefinition of system \mathcal{B} : in addition to the new conteNt and connection shown in Fig. 6, a conteNt q_{13} is introduced adding $R_{13}^2 = (R_1^2, R_3^2)$ and $R_{13}^3 = (R_1^3, R_3^3)$ to the two respective bunches (for c_2 and c_3).

or more bunches may very well intersect with the same two or more connections. In the system \mathcal{B} , the conteNts q_1 and q_2 are represented by R_1^1, R_2^1 in the conteXt c_1 (a short way of saying “in the bunch labeled by the conteXt c_1 ”), and the same q_1 and q_2 are represented by R_1^2, R_2^2 in the conteXt c_2 . This may make it desirable (but by no means necessary) to introduce a new conteNt q_{12} and the corresponding connection formed by $R_{12}^1 = (R_1^1, R_2^1)$ and $R_{12}^2 = (R_1^2, R_2^2)$. In our conceptual framework this means replacing \mathcal{B} with another system, shown in Fig. 6. The new system has a different set of conteNts and its contextuality analysis generally will not coincide with that of the system \mathcal{B} .

One can analogously introduce a new connection formed by $R_{13}^2 = (R_1^2, R_3^2)$ and $R_{13}^3 = (R_1^3, R_3^3)$ to bridge the bunches for the conteXts c_2 and c_3 . One can combine this connection with the one for q_{12} . (One could even add a new variable $R_{123} = (R_1^2, R_2^2, R_3^2)$ to the second bunch, for c_2 , but a connection consisting of a single bunch never affects contextuality analysis and can be dropped.) The approach to contextuality presented in this paper allows for any such modifications. However, we do not consider them obligatory, and in some cases, as in the system \mathcal{A} of Fig. 1, they may be considered too restrictive (see the discussion in Section 1.8).

1.7. The intuition for (non)contextuality

The main idea can be intuitively presented as follows. We have defined a c-c system as a set of conteXt-representing bunches together with connections between these bunches that reflect commonality of conteNts. It is equally possible, however, to view a c-c system as a set of conteNt-representing connections related to each other by bunches that reflect commonality of conteXts. Thus, the system \mathcal{B} depicted in Fig. 2 consists of the three connections, the elements of each of which are pairwise stochastically unrelated random variables. However, the element R_1^1 of the first connection is stochastically related to the element R_2^2 of the second connection because they share a conteXt; and analogously for the other two conteXts.

We distinguish two forms of the dependence of random variables on their conteXts. One of them is “*contextuality proper*,” the other one we call “*direct influences*.” Let us begin with the latter. Direct influences are reflected in the differences, if any, between the distributions of the elements of the same connection. For instance, if R_1^1 and R_1^2 have different distributions, then the change of the conteXt from c_1 to c_2 directly influences the random variable representing the conteNt q_1 . Direct influences are important, but they are of the same nature as the dependence of random variables on their conteNt. Thus, if the distribution of responses to the question about bees changes depending on what movie has been previously watched, the influence of the movie on the response is not any more puzzling than the influence of the question itself. We prefer not to use the term “contextuality” for such forms of conteXt-dependence. Instead we describe them by saying that the system of random variables is *inconsistently connected* if some of the elements of some of the connections in it have different distributions. If in a system all elements of any connection have the same distribution, then we call the system *consistently connected*.

There is a simple and universal (applicable to all systems) way to measure the degree of direct influences within each connection. Although the connections in a c-c system consist of pairwise stochastically unrelated random variables, nothing prevents one from thinking *counterfactually* of what their joint distribution could be *if they were* jointly distributed. (The reader should for now suspend criticism, as later on we will define this counterfactual reasoning rigorously.)

Let us take, e.g., the connection (R_1^1, R_1^2, R_1^3) in the system \mathcal{B} depicted in Fig. 2. If these random variables were jointly distributed, then we would be able to compute the probability with which they assume identical values, $\Pr[R_1^1 = R_1^2 = R_1^3]$. One may ask: among all possible “imaginary” joint distributions of (R_1^1, R_1^2, R_1^3) (in which all three of them retain their individual distributions), what is the maximal possible value for $\Pr[R_1^1 = R_1^2 = R_1^3]$? As it turns out, this maximal probability is well-defined and uniquely determinable: let us denote it \max_1 . We define \max_2 for the con-

nection (R_2^1, R_2^2) as the maximal “imaginary” value for $\Pr[R_2^1 = R_2^2]$ given the individual distributions of R_2^1 and R_2^2 ; and we define \max_3 for the connection (R_3^2, R_3^3) analogously. These maximal probabilities of coincidence can be viewed as reflecting the degree of direct influences of contexts upon the random variables: e.g., $\max_1 = 1$ if and only if the distributions of all three random variables in (R_1^1, R_1^2, R_1^3) are the same; and $\max_1 = 0$ if and only if the direct influences are so prominent that the equality $R_1^1 = R_1^2 = R_1^3$ becomes “unimaginable”: they cannot occur in any of the imagined joint distributions because the supports of the three variable (the subsets of possible values that have nonzero probability masses) do not have elements in common.

Now we come to “contextuality proper.” The maximal probabilities just discussed are computed for each connection taken separately, without taking into account the bunches that reflect the commonality of contexts across the connections. The question arises: are these maximal probability values,

$$\begin{aligned}\Pr[R_1^1 = R_1^2 = R_1^3] &= \max_1, \\ \Pr[R_2^1 = R_2^2] &= \max_2, \\ \Pr[R_3^2 = R_3^3] &= \max_3,\end{aligned}\tag{1}$$

compatible with the observed bunches of the system? In other words, can one achieve these maximal (imaginary) probabilities in all three connections simultaneously if one takes into account all the known (not imagined) joint distributions in the bunches of the system? If the answer is affirmative, then we can say that the knowledge of the bunches representing different contexts adds nothing to what we already know of the direct influences by having considered the connections separately — we call such a system *noncontextual*. If the answer is negative, however, then the contexts do influence the random variables beyond any direct influences they exert on them — the system is *contextual*.¹

1.8. (Non)contextuality of consistently connected systems

A system that exhibits no direct influences at all (i.e., is consistently connected) may very well be contextual. In a consistently connected version of our system \mathcal{B} the three maximal probability values will all be 1, and a system

will be contextual if the “imaginary” equalities

$$\begin{aligned}\Pr[R_1^1 = R_1^2 = R_1^3] &= 1, \\ \Pr[R_2^1 = R_2^2] &= 1, \\ \Pr[R_3^2 = R_3^3] &= 1,\end{aligned}\tag{2}$$

are incompatible with the observed bunches of the system. This will be the case when one can say, by abuse of language, that the system is contextual because the elements in each of its connections cannot be viewed as being essentially one and the same random variable.

Consistent connectedness, in special forms, is known under a variety of other names. In psychology, within the framework of so-called *selective influences*, the term describing consistent connectedness is “marginal selectivity” (Townsend & Schweickert, 1989). In quantum physics it is often called “no-signaling” property, especially when dealing with the EPR-type paradigms discussed in Section 5 (Popescu & Rohrlich, 1994; Masanes, Acín, & Gisin, 2006), or somewhat more generally, “no-disturbance” property (Kurzynski, Cabello, & Kaszlikowski, 2014). Cereceda (2000) lists several other terms.

Many scholars, especially in quantum mechanics, have considered contextuality for consistently connected systems only (e.g., Dzhafarov & Kujala, 2014c; Fine, 1982; Kurzynski, Ramanathan, Kaszlikowski, 2012; Kurzynski et al., 2014). The same is true for the contextuality theory of Abramsky and colleagues when it is applied to systems of random variables (Abramsky & Brandenburger, 2011; Abramsky et al., 2015). As a rule, however, consistent connectedness is considered in a *strong version*, wherein a consistently connected system should satisfy the following property: in any two bunches R^1, R^2 that share a set of contexts q_1, \dots, q_k , the corresponding sets of random variables (R_1^1, \dots, R_k^1) and (R_1^2, \dots, R_k^2) have one and the same joint distribution. In the theory of selective influences this property, or requirement, is called “*complete marginal selectivity*” (Dzhafarov, 2003).

When applied to the system \mathcal{B} in Fig. 2, the strong form of consistent connectedness means that, in addition to the same distribution of the random variables in each of the three connections of \mathcal{B} , we also posit the same distribution for (R_1^1, R_2^1) and (R_1^2, R_2^2) and for (R_1^2, R_3^2) and (R_1^3, R_3^3) . It is easy to see that this amounts to replacing the system \mathcal{B} with the redefined system \mathcal{B}' shown in Fig. 7, and assuming that it is consistently connected in the “ordinary” sense. The CbD theory allows for both \mathcal{B} and \mathcal{B}' to represent one and the same empirical situation, the choice between them being outside the scope of the theory. Therefore the notion of consistent connectedness in this paper includes the strong version thereof as a special case.

The difference between the strong and weaker forms of consistent connectedness is especially transparent if we consider the system \mathcal{A} of our opening example (Fig. 1). Its consistent connectedness means that the distribution

¹ In reference to footnote 10 below, in the newer version of CbD (Dzhafarov & Kujala, 2016a,b), in the case of more than two random variables, as in (R_1^1, R_1^2, R_1^3) , the maximum probability should be considered not only for $R_1^1 = R_1^2 = R_1^3$ but also for each of $R_1^1 = R_1^2, R_1^2 = R_1^3$, and $R_1^1 = R_1^3$.

R_q^c	$R_{q'}^c$	$(R_q^c, R_{q'}^c)$	c	$(R_q^c, R_{q'}^c)$	c
$R_q^{c'}$	$R_{q'}^{c'}$	$(R_q^{c'}, R_{q'}^{c'})$	c'	$(R_q^{c'}, R_{q'}^{c'})$	c'
q	q'	q''	$\boxed{\mathcal{A}^*}$	q''	$\boxed{\mathcal{A}^{**}}$

Figure 8. Two modifications of system \mathcal{A} in our opening example that are required to consider this original system consistently connected in the strong sense (“non-signaling,” or “complete marginal selectivity”). The two modifications are equivalent if they are consistently connected in the “ordinary” sense.

of responses to a given question, q or q' , is the same irrespective of the context, c or c' . The correlations between the two responses, however, may very well be different in the two contexts. If this is the case, the consistently connected system \mathcal{A} can be shown to be contextual (see Section 5). By contrast, if one assumes the strong form of consistent connectedness, the system \mathcal{A} is replaced with the system \mathcal{A}^* shown in Fig. 8, consistently connected in the “ordinary” sense. This system is trivially noncontextual, as its two bunches have the same distribution. In Fig. 8 this system is shown together with the system \mathcal{A}^{**} in which the first two columns of the c - c matrix representing \mathcal{A}^* are dropped as redundant. Note, however, that the systems \mathcal{A}^* and \mathcal{A}^{**} are not equivalent if they are not consistently connected: the single-connection system \mathcal{A}^{**} , as should be clear from Section 1.7, is always noncontextual, whereas the system \mathcal{A}^* may very well be contextual.

2. CONTEXTS AND CONTENTS: A FORMAL TREATMENT

Here, we present the basic conceptual set-up of our theory: a random variable (confined to categorical random variables), jointly distributed random variables (confined to finite sets thereof), functions of random variables, and systems of random variables, with bunches and connections. The reader who is not interested in a systematic introduction may just skim through Sections 2.4 and 2.5 and proceed to Section 3.

Our view of random variables and relations among them is “discourse-relative,” in the sense that the existence of these variables and relations depends on what other random variables are “in play.”

2.1. Categorical random variables

We begin with a class \mathbf{E} of (categorical) random variables that we consider “existing” (or “defined,” or “introducible,” etc.). We need not be concerned with the cardinality of \mathbf{E} as in this paper we will always deal with

finite subsets thereof.² A random variable X is a pair

$$X = (\text{id}X, \text{di}X), \quad (3)$$

where $\text{id}X$ is its unique *identity label* (within the class \mathbf{E}), whereas $\text{di}X$ (to be read as a single symbol) is its *distribution*. The latter in turn is defined as a function

$$\text{di}X : V_X \rightarrow [0, 1], \quad (4)$$

where V_X is a *finite set* (called the *set of possible values* of the random variable X), and

$$\sum_{v \in V_X} \text{di}X(v) = 1. \quad (5)$$

The value $\text{di}X(v)$ for any $v \in V_X$ is referred to as the *probability mass* of X at its value v . For any subset W of V_X we define the probability of $X \in W$ as

$$\Pr[X \in W] = \sum_{v \in W} \text{di}X(v). \quad (6)$$

In particular, for $v \in V_X$,

$$\Pr[X \in \{v\}] = \text{di}X(v), \quad (7)$$

and we may also write $\Pr[X = v]$ instead of $\Pr[X \in \{v\}]$.

Note that we impose no restrictions on the nature of the values v , only that their set V_X is finite. In particular, if V_1, \dots, V_n are finite sets, then a random variable $Z \in \mathbf{E}$ with a distribution

$$\text{di}Z : V_1 \times \dots \times V_n \rightarrow [0, 1] \quad (8)$$

is a categorical random variable. It can be denoted $Z = (X_1, \dots, X_n)$, where X_i is called the i th component (or the i th 1-marginal) of Z , with the distribution defined by

$$\sum_{\substack{(v_1, \dots, v_i, \dots, v_n) \\ \in V_1 \times \dots \times V_{i-1} \\ \times \{v_i\} \\ \times V_{i+1} \times \dots \times V_n}} \text{di}Z(v_1, \dots, v_n) = \text{di}X_i(v_i), \quad (9)$$

for any $v_i \in V_i$. The summation in this formula is across all possible n -tuples (v_1, \dots, v_n) with the value of v_i being fixed.

Definition 2.1. We will say that X_1, \dots, X_n in \mathbf{E} are *jointly distributed* if they are 1-marginals of some $Z = (X_1, \dots, X_n)$ in \mathbf{E} . The random variable Z then can be called a vector (sequence, n -tuple), of *jointly distributed* X_1, \dots, X_n . If X_1, \dots, X_n are not jointly distributed, they are *stochastically unrelated* (in \mathbf{E}).

² The cardinality need not even be defined, as we consider \mathbf{E} a class rather than a set.

Note that according to this definition, X_1, \dots, X_n in \mathbf{E} are not jointly distributed if \mathbf{E} does not contain (X_1, \dots, X_n) , even though one can always conceive of a joint distribution for them. This reflects our interpretation of \mathbf{E} as the class of the variables that “exist” (rather than just “imagined,” as discussed in Section 1.7).

For any subsequence (i_1, \dots, i_k) of $(1, \dots, n)$ one can compute the corresponding k -marginal of Z . Without loss of generality, let $(i_1, \dots, i_k) = (1, \dots, k)$. Then the k -marginal $Y = (X_1, \dots, X_k)$ has the distribution defined by

$$\sum_{\substack{(v_1, \dots, v_k, v_{k+1}, \dots, v_n) \\ \in \{v_1\} \times \dots \times \{v_k\} \\ \times V_{k+1} \times \dots \times V_{X_n}}} \text{di}Z(v_1, \dots, v_n) = \text{di}Y(v_1, \dots, v_k), \quad (10)$$

for any $(v_1, \dots, v_k) \in V_1 \times \dots \times V_k$. The summation in this formula is across all possible n -tuples (v_1, \dots, v_n) with the values of v_1, \dots, v_k being fixed. This distribution of the k -marginal Y is referred to as a k -marginal distribution.

2.2. Functions of random variables

Let $X \in \mathbf{E}$ be a random variable with the distribution $\text{di}X : V_X \rightarrow [0, 1]$, and let $f : V_X \rightarrow f(V_X)$ be some function. The function $f(X)$ of a random variable X is a random variable Y such that X and Y are 1-marginals of some random variable $Z = (X, Y)$ with the distribution $\text{di}Z : V_X \times f(V_X) \rightarrow [0, 1]$ defined by

$$\text{di}Z(v, w) = \begin{cases} \text{di}X(v) & \text{if } w = f(v) \\ 0 & \text{if otherwise} \end{cases}. \quad (11)$$

It follows that the distribution of Y as a 1-marginal of Z is defined by

$$\text{di}Y(w) = \sum_{v \in f^{-1}(\{w\})} \text{di}X(v), \quad (12)$$

for any $w \in f(V_X)$.

We stipulate as the main property of the class \mathbf{E} that *any function of X in \mathbf{E} belongs to \mathbf{E}* . This property together with the definition of 1-marginals implies that $Z = (X, f(X))$ belongs to \mathbf{E} , i.e., X and $f(X)$ are jointly distributed.

If $Y_1 = f_1(X)$ and $Y_2 = f_2(X)$, we can consider (f_1, f_2) as a function f mapping V_X into $f_1(V_X) \times f_2(V_X)$. Then $Y = (Y_1, Y_2)$ being a function of X is merely a special case of the situation considered above. Its meaning is that X and (Y_1, Y_2) are 1-marginals of some random variable $Z = (X, (Y_1, Y_2))$ (that belongs to

\mathbf{E}) whose distribution is defined by

$$\text{di}Z(v, (w_1, w_2)) = \begin{cases} \text{di}X(v) & \text{if } (w_1, w_2) = f(v) = (f_1(v), f_2(v)) \\ 0 & \text{if otherwise} \end{cases}. \quad (13)$$

The (1-marginal) distribution of $Y = (Y_1, Y_2)$ is defined by

$$\begin{aligned} \text{di}Y(w_1, w_2) &= \sum_{v \in f^{-1}(\{(w_1, w_2)\})} \text{di}X(v) \\ &= \sum_{v \in f_1^{-1}(\{w_1\}) \cap f_2^{-1}(\{w_2\})} \text{di}X(v). \end{aligned} \quad (14)$$

The random variables Y_1 and Y_2 themselves are 1-marginals of $Y = (Y_1, Y_2)$ just defined. Indeed, the separate distribution of Y_1 computed in accordance with (12) is

$$\text{di}Y_1(w_1) = \sum_{v \in f_1^{-1}(\{w_1\})} \text{di}X(v), \quad (15)$$

for any $w_1 \in f_1(V_X)$. We get the same formula from (14) by applying to it the formula for computing 1-marginals, (9):

$$\begin{aligned} \text{di}Y_1(w_1) &= \sum_{w_2} \sum_{v \in f_1^{-1}(\{w_1\}) \cap f_2^{-1}(\{w_2\})} \text{di}X(v) \\ &= \sum_{v \in f_1^{-1}(\{w_1\}) \cap (\bigcup_{w_2} f_2^{-1}(\{w_2\}))} \text{di}X(v), \\ &= \sum_{v \in f_1^{-1}(\{w_1\})} \text{di}X(v), \end{aligned} \quad (16)$$

because the union of the sets $f_2^{-1}(\{w_2\})$ across all values of $w_2 \in f_2(V_X)$ is the entire set V_X . Analogous reasoning applies to Y_2 .

This shows that Y_1 and Y_2 defined as functions of some $X \in \mathbf{E}$ are jointly distributed in the sense of Definition 2.1: they are 1-marginals of some $Y = (Y_1, Y_2) \in \mathbf{E}$. It is easy to show that the converse holds true as well: if Y_1 and Y_2 in \mathbf{E} are jointly distributed, then they are functions of one and the same random variable that belongs to \mathbf{E} . Indeed, in accordance with Definition 2.1, they are 1-marginals of some $Y = (Y_1, Y_2) \in \mathbf{E}$. But then Y_1 and Y_2 are functions of this Y . Specifically, denoting the sets of possible values for Y_1, Y_2 by W_1, W_2 , respectively, we have $Y_1 = f_1(Y)$, where

$$f_1 : W_1 \times W_2 \rightarrow W_1 \quad (17)$$

is defined by $f_1(w_1, w_2) = w_1$ (a projection function). The computations of the distribution of Y_1 in accordance with (9) coincides with that in accordance with (12),

$$\text{di}Y_1(w_1) = \sum_{w_2 \in W_2} \text{di}Y(w_1, w_2) = \sum_{v \in f_1^{-1}(\{w_1\})} \text{di}Y(v), \quad (18)$$

for any $w_1 \in W_1$. Analogous reasoning applies to Y_2 .

This result is trivially generalized to an arbitrary finite set of random variables.³

Theorem 2.2. *Random variables $X_1, \dots, X_n \in \mathbf{E}$ are jointly distributed if and only if they are representable as functions of one and the same random variable $X \in \mathbf{E}$.*

Note that this X may very well equal one of the X_1, \dots, X_n . More generally, one can add X to its functions X_1, \dots, X_n to create a jointly distributed set X, X_1, \dots, X_n in which all elements are, obviously, functions of one of its elements. Note also that if $X_i = f_i(X)$, $i = 1, \dots, n$, then $f = (f_1, \dots, f_n)$ is a function, and we can equivalently reformulate Theorem 2.2 as saying that a vector of random variables (X_1, \dots, X_n) is a random variable (by definition, with jointly distributed components) if and only if it is a function of some random variable X .

In spite of its simplicity, Theorem 2.2 was discovered, in various special forms, only in the 1980s (Suppes & Zanotti, 1981; Fine, 1982). It has a direct bearing on the problem of “hidden variables” in quantum mechanics: given a set of random variables, is there a random entity of which these random variables are functions? To formulate this problem rigorously and to enable the use of Theorem 2.2 for solving it we will need the notion of a coupling, introduced below (Section 3.1).

2.3. Two meanings of equality of random variables

The following remark may prevent possible confusions. Given a random variable Z and a measurable set E , the expression $Z \in E$ clearly does not mean that Z as a random variable (with its identity $\text{id}Z$ and distribution $\text{di}Z$) is an element of E . Rather this expression is a way of saying that we are considering an event E in the measure space $\text{di}Z = (S_Z, \Sigma_Z, \mu_Z)$ associated with a random variable Z . Thus, $\Pr[Z \in E]$ is $\mu_Z(E)$. As a special case, given jointly distributed X, Y , the expression $X = Y$ is merely a shortcut for $(X, Y) \in W$, where $W = \{(v_1, v_2) \in S_X \times S_Y : v_1 = v_2\}$. This meaning of equality should not be confused with another meaning: $X = Y$ can also mean that these two symbols refer to one and the same random variable, so that $\text{id}X = \text{id}Y$ and (consequently) $\text{di}X = \text{di}Y$. We think that the meaning of $X = Y$ in this paper is always clear from the context (now using this word without capital X).

2.4. Base sets of random variables

How does one construct the class \mathbf{E} ? For instance, with X_1, \dots, X_n all in \mathbf{E} , how do we know whether

they are jointly distributed, i.e., whether \mathbf{E} contains a $Z = (X_1, \dots, X_n)$? Can we simply declare that any random variables X_1, \dots, X_n are jointly distributed? The answer to the last question is negative: to be able to model empirical phenomena one needs to keep the meaning of joint distribution tied to the empirical meaning of “co-occurrence” — which means that joint distribution cannot be imposed arbitrarily.

To make all of this clear, let us construct the class \mathbf{E} of “existing” random variables systematically. The construction is simple: we introduce a nonempty *base set* \mathbf{R} of (categorical) random variables (in this paper we assume this set to be finite, but this need not be so generally), and we posit that

- (P1): a random variable belongs to \mathbf{E} if and only if it is a function of any one of the elements of \mathbf{R} ;
- (P2): no random variable in \mathbf{E} is a function of two distinct elements of \mathbf{R} .

The constraints (P1-P2) ensure that no two random variables existing in the sense of belonging to \mathbf{E} may have a joint distribution unless they are functions of one and the same element of \mathbf{R} . Indeed, let some transformations $\alpha(A)$ and $\beta(B)$ have a joint distribution, for $A, B \in \mathbf{R}$. Then a random variable $(\alpha(A), \beta(B))$ exists, which means that this pair is a function of some $C \in \mathbf{R}$. But then $\alpha(A)$ is a function of A and C , whence $A = C$, and $\beta(B)$ is a function of both B and C , whence $B = C = A$. (In reference to Section 2.3, the equalities here are used in the sense of “one and the same random variable.”)

Instead of “ X belongs to \mathbf{E} ” we can also say “ X exists with respect to \mathbf{R} .” This is preferable if one deals with different base sets \mathbf{R} inducing different classes \mathbf{E} , as we do in the subsequent sections.

Consider an example: let \mathbf{R} consist of the four random variables

$$X = (X_1, X_2, X_3), Y = (Y_1, Y_2), Z, U = (U_1, U_2).$$

These random variables are declared to exist, and then so are functions of these random variables. Thus, X_2 exists because X exists and X_2 is its function (second projection). Analogously, if the values of Y_1, Y_2 are numerical, the variable $Y_1 + Y_2$ exists. However, no component of one of the four random variables, say, X_2 , is jointly distributed with any component of another, say, U_1 , and no function $f(U_1, X_2)$ is a random variable (its distribution is undefined). By the same logic, no two different vectors in \mathbf{R} can share a component: if they did, this component would be a function of both of them, contravening (P2).

2.5. Systems of random variables

The example of \mathbf{R} at the end of the previous section is in fact how we introduce our main object: *context-conteNt systems of random variables*.

³ In fact it holds for any set of any random entities (Dzhafarov & Kujala, 2010), but our focus in this paper is on finite sets of categorical random variables.

Definition 2.3. Let R be a base set of (categorical) random variables each element of which, called a *bunch*, is a vector of random variables. Let U_R be the union of all components of these bunches. A *conteXt-conteNt (c-c) system* \mathcal{R} of random variables based on R is created by endowing R with a partition of U_R into subsets called *connections* and satisfying the following two properties:

(*intersection property*) a bunch and a connection do not have more than one component of U_R in common; and

(*comparability property*) elements of a connection have the same set of possible values.⁴

Recall that partitioning of a set means creating a set of pairwise disjoint subsets whose union is the entire set. Note that due to the intersection property in Definition 2.3, any two elements of a connection are stochastically unrelated (they are 1-marginals of different bunches). Note also that due to the comparability property the elements of a connection may (but generally do not) have the same distribution.

Let the bunches of the c-c system be enumerated $1, \dots, n$, and the connections be enumerated $1, \dots, m$. Due to the intersection property in Definition 2.3, any random variable in the set U_R of a c-c system \mathcal{R} can be uniquely identified by the labels of the bunch and of the connection it belongs to. These labels (or some symbols in a one-to-one correspondence with them) are referred to as *conteXts* (labels for bunches) and *conteNts* (labels for connections). As we see, in the formal theory bunches and connection define rather than are defined by the conteXts and conteNts, respectively. It is the other way around in empirical applications (see the introductory section), where our understanding of what constitutes a given conteNt under different conteXts guides the creation of the bunches and connections.

The unique labeling of the random variables by the conteXts and conteNts means that any c-c system can be presented in the form already familiar to us from the introduction: a conteXt-conteNt (c-c) matrix. An example of a c-c system presented in the form of a c-c matrix is given in Fig. 2. The initial base set of random variables is

$$R = \left\{ \begin{array}{l} R^1 = (R_1^1, R_2^1) \\ R^2 = (R_1^2, R_2^2, R_3^2) \\ R^3 = (R_1^3, R_3^3) \end{array} \right\}, \quad (19)$$

the union set is

$$U_R = \{R_1^1, R_2^1, R_1^2, R_2^2, R_3^2, R_1^3, R_3^3\} \quad (20)$$

with the lower indexes already chosen in view of the partitioning into connections,

$$\left\{ \begin{array}{l} (R_1^1, R_2^1, R_3^1) \\ (R_2^1, R_2^2) \\ (R_3^2, R_3^3) \end{array} \right\}. \quad (21)$$

The intersection property in Definition 2.3 is critical: if a conteXt and a connection could have more than one random variable in common, both the double-indexing of the random variables by conteXts and conteNts and the subsequent contextuality analysis of the system would be impossible.

2.6. Kolmogorovian Probability Theory and Contextuality-by-Default

In this section we briefly discuss the relationship between KPT and CbD. This discussion is not needed for understanding the subsequent sections. We will assume the reader's familiarity with the basics of measure theory.

The definition of (categorical) random variables in KPT is as follows. Let (S, Σ, μ) be a domain probability space, and let (V_X, Σ_X) be a codomain measurable space, with V_X a finite set and Σ_X usually (and here) defined as its power set. A random variable X is a function $S \rightarrow V_X$ such that $X^{-1}(\{v\}) \in \Sigma$, for any $v \in V_X$. The probability mass $p_X(v)$ is defined as $\mu(X^{-1}(\{v\}))$, and for any subset $V \subset V_X$, the probability of X falling in V is computed as

$$\Pr[X \in V] = \mu(X^{-1}(V)) = \sum_{v \in V} p_X(v). \quad (22)$$

We call (S, Σ, μ) the *sample space*⁵ for X .

The great conceptual convenience of KPT is that the joint distribution of two random variables taken as two functions defined on the same sample space is uniquely determined by these two functions: if X is as above and Y is another random variable, then its joint distribution with X above is defined by

$$p_{XY}(v, w) = \mu(X^{-1}(\{v\}) \cap Y^{-1}(\{w\})), \quad (23)$$

for any $(v, w) \in V_X \times V_Y$.

In CbD, random variables are considered only with respect to a specified base set. A random variable exists if it is a function of one and only one of the elements

⁴ In a more general treatment this translates into the same set and the same sigma algebra of events.

⁵ It seems common to use this term for the set S alone; but the term “space” in mathematics means a set with some structure imposed on it, and the structure here is the sigma algebra and the measure. We prefer therefore to use the term “sample space” for the entire domain probability space. S alone can be referred to as the *sample set* for X .

of this base set; and functions of different base random variables are considered stochastically unrelated. Is this picture compatible with KPT? We think it is, provided KPT is not naively thought of as positing the existence of a single sample space for all imaginable random variables. Such a view can be shown to be mathematically flawed (Dzhafarov & Kujala, 2014a-b).

Every sample space (S, Σ, μ) corresponds to a random variable Z defined as the identity mapping $S \rightarrow S$ from (S, Σ, μ) to (S, Σ) , and then every random variable defined on (S, Σ, μ) is representable as a transformation of Z . If we consider a set of sample spaces unrelated to each other, then the corresponding identity functions form a base set of random variables, and what we get is essentially the same picture as in CbD.

We need one qualification though: even if all the functions considered are categorical random variables, the base set itself need not be a finite set of categorical random variables, as it is in Section 2.5. This is not, however, a restriction inherent in CbD but the choice we have made in this paper. A finite number of categorical base variables are sufficient if one only considers a finite set of functions thereof, which is the case we deal with.

3. CONTEXTUALITY ANALYSIS

In this section we give the definitions and introduce the conceptual apparatus involved in determining whether a c-c system is contextual or noncontextual.

3.1. Probabilistic couplings

Imagining joint distributions for things that are not jointly distributed, as it was presented in Section 1.7, is not rigorous mathematics. The latter requires that we use the mathematical tool of (*probabilistic*) *couplings*.

Definition 3.1. A coupling of a set of random variables X_1, \dots, X_n is a random variable (Y_1, \dots, Y_n) (with jointly distributed components) such that Y_i has the same distribution as X_i , for all $i = 1, \dots, n$.

As an illustration, let X_1 and X_2 be distributed as

$$\text{pr. mass} \begin{array}{|c|c|c|} \hline X_1 = 1 & X_1 = 2 & X_1 = 3 \\ \hline 0.3 & 0.3 & 0.4 \\ \hline \end{array}$$

and

$$\text{pr. mass} \begin{array}{|c|c|} \hline X_2 = 1 & X_2 = 2 \\ \hline 0.7 & 0.3 \\ \hline \end{array}.$$

Then (Y_1, Y_2) with the distribution

$$\begin{array}{|c|c|c|c|} \hline & Y_1 = 1 & Y_1 = 2 & Y_1 = 3 \\ \hline Y_2 = 1 & 0.3 & 0.2 & 0.2 \\ Y_2 = 2 & 0 & 0.1 & 0.2 \\ \hline & 0.3 & 0.3 & 0.4 \\ \hline \end{array} 0.7$$

is a coupling for X_1 and X_2 . And so is (Y'_1, Y'_2) with the distribution

$$\begin{array}{|c|c|c|c|} \hline & Y'_1 = 1 & Y'_1 = 2 & Y'_1 = 3 \\ \hline Y'_2 = 1 & 0.3 & 0 & 0.4 \\ Y'_2 = 2 & 0 & 0.3 & 0 \\ \hline & 0.3 & 0.3 & 0.4 \\ \hline \end{array} 0.3.$$

Generally, the number of couplings of a given set of random variables is infinite.

In our paper couplings are constructed in two ways only: either for connections in a c-c system, taken separately, or for the entire set of bunches in the system. In relation to the system \mathcal{B} in Fig. 2 and (19), a coupling for the connection (R_1^1, R_1^2, R_1^3) is a triple (T_1^1, T_1^2, T_1^3) such that T_1^j and R_1^j have the same distribution, for $j = 1, 2, 3$; and analogously for the other two connections.

The set of the three bunches (R^1, R^2, R^3) in (19) is coupled by $S = (S^1, S^2, S^3)$ where

$$\left\{ \begin{array}{l} S^1 = (S_1^1, S_2^1) \\ S^2 = (S_1^2, S_2^2, S_3^2) \\ S^3 = (S_1^3, S_3^3) \end{array} \right\} \quad (24)$$

such that S^j and R^j have the same distribution, for $j = 1, 2, 3$.

In the following we will freely use phrases indicating that a coupling for some random variables “exists,” or “can be constructed,” or that these random variables “can be coupled.” Note, however, that the couplings do not “exist” with respect to the base set of random variables formed by the bunches of a c-c system, as no coupling of the bunches can be presented as a function of just one of these bunches. If the bunches are assumed to have links to empirical observations, then the couplings can be said to have no empirical meaning. A coupling forms a base set of its own, consisting of itself. Its marginals (or *subcouplings*) corresponding to the bunches of the c-c system do “exist” with respect to this new base set, as they are functions of its only element. However, the bunches of the c-c system themselves do not “exist” with respect to the base set formed by this coupling. One can add the coupling $S = (S^1, S^2, S^3)$ to the set (R^1, R^2, R^3) of the three bunches of our system \mathcal{B} as a fourth element of a new base set, stochastically unrelated to the bunches.

3.2. “Flattening” convention

Let us adopt the following simplifying convention in regard to couplings (and more generally, vectors of jointly distributed variables): a vector of jointly distributed random variables (A^1, \dots, A^n) in which $A^i = (A_{k_1}^i, \dots, A_{k_i}^i)$, for each $i = 1, \dots, n$, is considered equivalent (replaceable by) the vector

$$(A_1^1, \dots, A_{k_1}^1, \dots, A_1^n, \dots, A_{k_n}^n). \quad (25)$$

As each of the random variables is assumed to be uniquely indexed (in our analysis, double-indexed), the order in which they are shown in any given vector is usually arbitrary. As an example, a coupling $S = (S^1, S^2, S^3)$ of the three bunches $\mathcal{R} = (R^1, R^2, R^3)$ in (19), written in extenso, is

$$S = ((S_1^1, S_2^1), (S_1^2, S_2^2, S_3^2), (S_1^3, S_3^3)) \quad (26)$$

with (S_1^1, S_2^1) distributed as the bunch (R_1^1, R_2^1) , etc. In accordance with our agreement, this coupling can be equivalently written as

$$S = (S_1^1, S_2^1, S_1^2, S_2^2, S_3^2, S_1^3, S_3^3), \quad (27)$$

in which (S_1^1, S_2^1) distributed as the bunch (R_1^1, R_2^1) , etc. The “flattening” convention makes it easier to compare couplings of a connection taken in isolation with the *subcoupling* of the coupling (27) corresponding to the same connection. Thus, for the connection (R_1^1, R_1^2, R_1^3) taken separately we can consider all possible couplings (T_1^1, T_1^2, T_1^3) and then compare them with the subcouplings (S_1^1, S_1^2, S_1^3) extracted as 3-marginals from all possible couplings (27). We will need such comparisons for determining whether the system in question is contextual.

3.3. Maximal couplings for connections

Definition 3.2. Let R_j^1, \dots, R_j^k be a connection (for a content q_j) in a c-c system. A coupling (T_j^1, \dots, T_j^k) of R_j^1, \dots, R_j^k is a *maximal coupling* if the value of

$$\Pr [T_j^1 = \dots = T_j^k]$$

is the largest possible among all couplings of R_j^1, \dots, R_j^k .

(In relation to Section 2.3, the equality here clearly is not the identity of the random variables but a description of an event associated with jointly distributed variables.) Theorem 3.3 below ensures that the maximum mentioned in the definition always exist. The notion of a maximal coupling is well-defined for arbitrary sets of arbitrary random variables (see Thorisson, 2000), but we will only need it for connections formed by categorical variables.⁶

⁶ It is a common mistake to think that this notion may be useless outside the class of categorical variables: thus, one might erroneously assume that if the distributions of T_j^1, \dots, T_j^k are absolutely continuous with respect to the Lebesgue measure on the set of reals, then $\Pr [T_j^1 = \dots = T_j^k]$ must be zero. In fact, this probability can be any number between 0 and 1, because the joint distribution of T_j^1, \dots, T_j^k within the intersection of their supports may very well be concentrated on the diagonal $T_j^1 = \dots = T_j^k$. In particular, if R_j^1, \dots, R_j^k are identically distributed, then, irrespective of this distribution, they have a maximal coupling (T_j^1, \dots, T_j^k) with $\Pr [T_j^1 = \dots = T_j^k] = 1$.

For any coupling (T_j^1, \dots, T_j^k) of (R_j^1, \dots, R_j^k) ,

$$\Pr [T_j^1 = \dots = T_j^k] = \sum_{v \in V} \Pr [T_j^1 = v, \dots, T_j^k = v], \quad (28)$$

where V is the set of possible values shared by the elements of the connection. This sum is maximized across all possible couplings if each of the summands on the right-hand side is maximized separately. The maximal possible value for $\Pr [T_j^1 = v, \dots, T_j^k = v]$ (with the individual distributions of Y_i being fixed) is

$$\max p_v = \min (\Pr [T_j^1 = v], \dots, \Pr [T_j^k = v]). \quad (29)$$

Indeed, the probability of a joint event can never exceed any of the probabilities of the component events. To prove that a maximal coupling exists for any connection, we need to show that every value of (T_j^1, \dots, T_j^k) can be assigned a probability so that, for all $v \in V$,

$$\Pr [T_j^1 = v, \dots, T_j^k = v] = \max p_v, \quad (30)$$

and

$$\sum_{\substack{v_1, \dots, v_{i-1}, \\ v_i = v, \\ v_{i+1}, \dots, v_k}} \Pr [T_j^1 = v_1, \dots, T_j^k = v_k] = \Pr [Y_i = v], \quad (31)$$

for any $i = 1, \dots, k$. A simple proof that this is always possible can be found in Thorisson (2000, pp. 7-8 and 104-107).

Theorem 3.3. A maximal coupling (T_j^1, \dots, T_j^k) can be constructed for any connection (R_j^1, \dots, R_j^k) in a c-c system, with

$$\begin{aligned} & \Pr [T_j^1 = \dots = T_j^k = v] \\ &= \min (\Pr [T_j^1 = v], \dots, \Pr [T_j^k = v]), \end{aligned} \quad (32)$$

for any v in the set V of possible values of (each of) R_j^1, \dots, R_j^k .

As an example, let the variables R_1^1, R_1^2, R_1^3 in the first connection of the system \mathcal{B} of Fig. 2 be binary, with the possible values 1 and 2. Let

$$\text{pr. mass} \begin{array}{|c|c|} \hline R_1^1 = 1 & R_1^1 = 2 \\ \hline 0.3 & 0.7 \\ \hline \end{array},$$

$$\text{pr. mass} \begin{array}{|c|c|} \hline R_1^2 = 1 & R_1^2 = 2 \\ \hline 0.4 & 0.6 \\ \hline \end{array},$$

$$\text{pr. mass} \begin{array}{|c|c|} \hline R_1^3 = 1 & R_1^3 = 2 \\ \hline 0.7 & 0.3 \\ \hline \end{array}.$$

R_1^1	R_2^1	\cdot	c_1
\cdot	R_2^2	R_3^2	c_2
R_1^3	\cdot	R_3^3	c_3
q_1	q_2	q_3	SZLG

Figure 9. The c-c matrix for a Suppes-Zanotti-Leggett-Garg-type system. This is a cyclic system of rank 3, in the terminology of Section 5. Each conteXt includes two conteNts, and each conteNt is included in two conteXts. All random variables are binary.

Then, in the maximal coupling (T_1^1, T_1^2, T_1^3) ,

$$\begin{aligned} \max p_1 &= \Pr [T_1^1 = T_1^2 = T_1^3 = 1] \\ &= \min (0.3, 0.4, 0.7) = 0.3, \\ \max p_2 &= \Pr [T_1^1 = T_1^2 = T_1^3 = 2] \\ &= \min (0.7, 0.6, 0.3) = 0.3. \end{aligned}$$

We can now assign probabilities to the rest of the values of (T_1^1, T_1^2, T_1^3) in an infinity of possible ways, e.g., as shown below,

$$\begin{array}{l} \text{value: } 111 \ 112 \ 121 \ 211 \ 221 \ 212 \ 122 \ 222 \\ \text{pr. mass } \boxed{0.3} \ \boxed{0} \ \boxed{0} \ \boxed{0.1} \ \boxed{0.3} \ \boxed{0} \ \boxed{0} \ \boxed{0.3} \end{array},$$

so that these joint probabilities yield the right values of $\Pr [T_1^j = 1] = \Pr [R_1^j = 1]$, for $j = 1, 2, 3$.

Consider another example, using the second connection in the system \mathcal{B} , (R_2^1, R_2^2) . Suppose that both these variables have the same distribution:

$$\begin{array}{l} \text{value} \quad 1 \quad 2 \quad 3 \\ \text{pr. mass } \boxed{0.3} \ \boxed{0.2} \ \boxed{0.5} \end{array},$$

in which case we can say that this connection is *consistent* (and, to remind, if this is the case for all connections, then the system is consistently connected). The maximal coupling (T_2^1, T_2^2) here has a uniquely determined distribution

$$\begin{array}{l} T_2^1 = 1 \quad T_2^1 = 2 \quad T_2^1 = 3 \\ T_2^2 = 1 \quad \boxed{0.3} \quad \boxed{0} \quad \boxed{0} \quad 0.3 \\ T_2^2 = 2 \quad \boxed{0} \quad \boxed{0.2} \quad \boxed{0} \quad 0.2, \\ T_2^2 = 3 \quad \boxed{0} \quad \boxed{0} \quad \boxed{0.5} \quad 0.5 \\ \quad \quad 0.3 \quad 0.2 \quad 0.5 \end{array}$$

with $\Pr [T_2^1 = T_2^2] = 1$.

3.4. Contextuality

Definition 3.4. A coupling for (the bunches of) a c-c system is *maximally connected* if its subcouplings corresponding to the connections of the system are maximal

couplings of these connections. If a system has a maximally connected coupling, it is *noncontextual*. Otherwise it is *contextual*.⁷

For motivation of this definition, see Section 1.7. Let us illustrate this definition using the system of binary random variables first considered, in abstract, by Suppes and Zanotti (1981) and then, as a paradigm in quantum mechanics, by Leggett and Garg (1985). The c-c matrix for this system is presented in Fig. 9. There are three conteNts here, any two of which are represented (measured, responded to) in one of three possible conteXts. Figure 10 (top panel) shows this system schematically: a set of three bunches stochastically unrelated to each other, and three connections “bridging” them. Since any of the six random variables in the system has two possible values, any coupling

$$S = (S_1^1, S_2^1, S_2^2, S_3^2, S_3^3, S_1^3) \quad (33)$$

of this system has 2^6 possible values, and its distribution is defined by assigning 2^6 probability masses to them.

It is difficult to see how one could show graphically that six random variables are jointly distributed. In the bottom panel of Fig. 10 this problem is solved by invoking Theorem 2.2, according to which the random variables in a coupling are all functions of one and the same, “hidden” random variable. We do not need to specify this random variable and the functions producing the components of a coupling explicitly. It is always possible, however, to choose this random variable to be the coupling S itself, and treat the random variables in the coupling as projection functions: S_1^1 is the first projection of S , S_2^1 is the second projection of S , etc.

The distribution of the six random variables in the coupling should, by definition, agree with the bunches of the system \mathcal{B} , each of which is uniquely characterized by three probabilities:

$$\begin{aligned} \Pr [R_1^1 = 1] &= p_1^1 = \Pr [S_1^1 = 1], \\ \Pr [R_2^1 = 1] &= p_2^1 = \Pr [S_2^1 = 1], \end{aligned} \quad (34)$$

$$\Pr [R_1^1 = 1, R_2^1 = 1] = p_{12} = \Pr [S_1^1 = 1, S_2^1 = 1]$$

for the bunch (R_1^1, R_2^1) , and analogously for the other two. There are generally an infinity of couplings that satisfy these equations.

Consider now the three connections of the system as three separate pairs of random variables (Fig. 11, top panel), and for each of them consider its coupling

⁷ In reference to footnote 10 below, in a newer version of CbD (Dzhafarov & Kujala, 2016a,b) (non)contextuality is defined in terms of multimaximal couplings, rather than merely maximal ones. This does not, however, affects the logic of CbD in any nontrivial way.

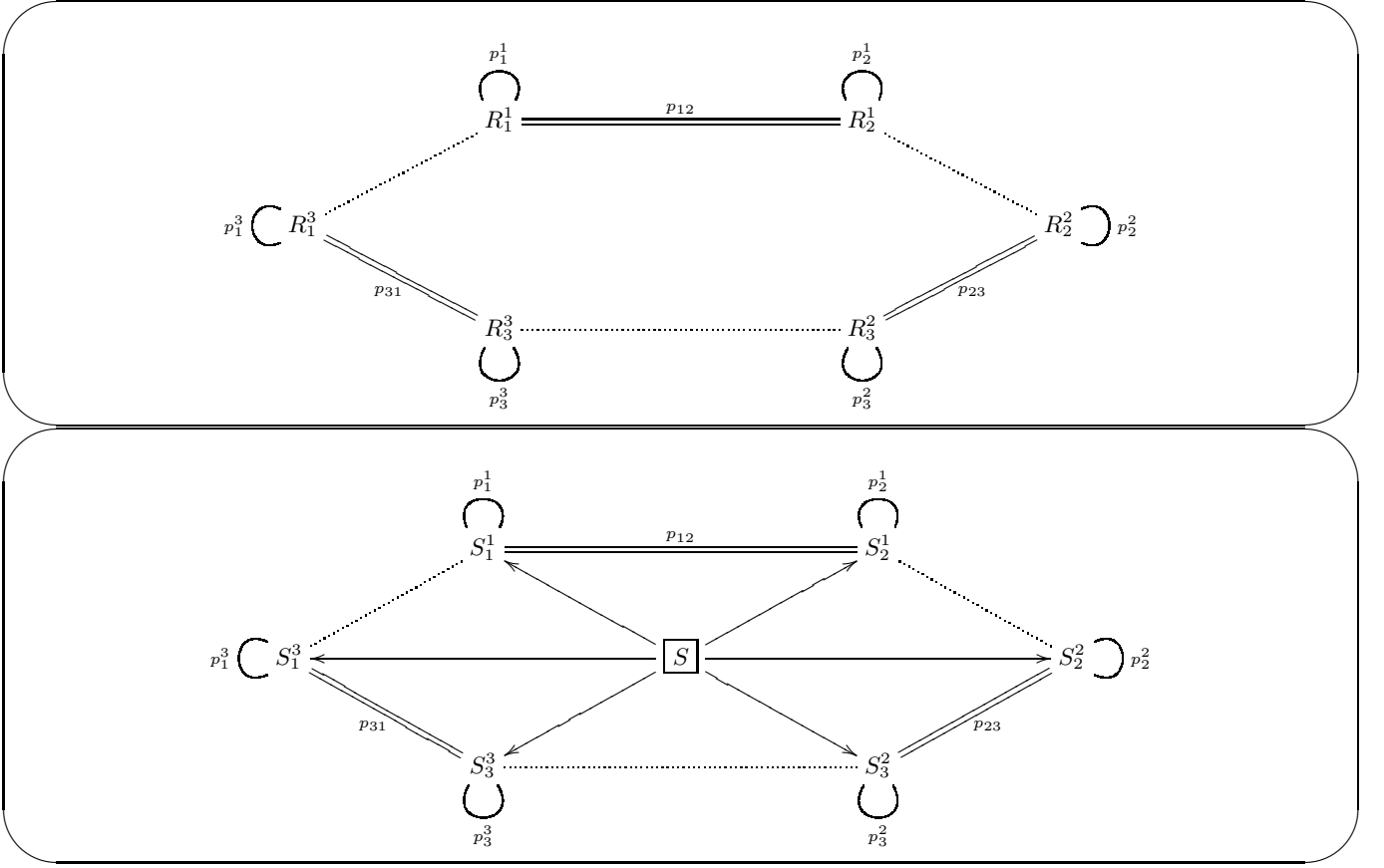


Figure 10. A schematic representation of a SZLG c-c system (top) and of its coupling (bottom). The coupling is shown as a set of random variables that are functions of some random variable S . The symbols p_i^j attached to R_i^j show the probability with which this variable equals 1; the symbols p_{kl} attached to double-lines show the joint probability with which the flanking variables equal 1. All these probabilities are preserved in the coupling because, by definition, the bunches of the system (the pairs of random variables connected by double-lines) have the same distribution as the corresponding subcouplings of the coupling. The elements of the connections of the system (the pairs of random variables connected by dotted lines) are stochastically unrelated, but the corresponding subcouplings of the coupling are jointly distributed, as 2-marginals of the coupling.

(Fig. 11, middle panel). The distributions of the elements of a connection are fixed, and its coupling should, by definition, preserve them. Thus, for the connection (R_1^1, R_1^3) ,

$$\begin{aligned} \Pr [R_1^1 = 1] &= p_1^1 = \Pr [T_1^1 = 1], \\ \Pr [R_1^3 = 1] &= p_1^3 = \Pr [T_1^3 = 1]. \end{aligned} \quad (35)$$

With p_1^1 and p_1^3 given, the distribution of the coupling (T_1^1, T_1^3) of (R_1^1, R_1^3) is determined by

$$\Pr [T_1^1 = 1, T_1^3 = 1] = p_1. \quad (36)$$

By Theorem 3.3, p_1 can be chosen so that $\Pr [T_1^1 = T_1^3]$ attains its maximal possible value, and this choice is

$$p_1 = \min(p_1^1, p_1^3). \quad (37)$$

We know that such a coupling is a maximal coupling of (R_1^1, R_1^3) , in this simple case, uniquely determined. We

choose values p_2 and p_3 for the maximal couplings of the remaining two connections analogously.

In accordance with Definition 3.4, the question now is whether these three values of the joint probabilities, p_1, p_2, p_3 , are compatible with the bunches of the system. Put differently, can one construct a maximally connected coupling shown in Fig. 11, bottom panel, a coupling in which all the probabilities shown are achieved together? The system is noncontextual if and only if the answer to this question is affirmative.

To see that it does not have to be affirmative, consider the example presented in Fig. 12. The maximally connected coupling does not exist because, in the bottom-panel diagram,

- (i) going clockwise from S_1^1 and using the transitivity of the relation “always equals,” we conclude that $\Pr [S_1^1 = S_3^3] = 1$;
- (ii) going counterclockwise from S_1^1 we see that $\Pr [S_1^1 = S_1^3] = 1$;

- (iii) but then $\Pr[S_1^3 = S_3^3]$ must also be 1, which it is not.

4. CONTEXTUALITY AS A LINEAR PROGRAMMING PROBLEM

Is there a general method for establishing contextuality or lack thereof in a given c-c system? It turns out that such a method exists, and insofar as finite sets of categorical random variables are involved, it is a simple linear programming method. A maximally connected coupling of a c-c system is uniquely associated with a certain underdetermined system of linear equations, and the c-c system is contextual if and only if this system of linear equations has no nonnegative solutions. The theory of these equations generalizes the Linear Feasibility Test described in Dzhaferov and Kujala (2012).

4.1. Notation and conventions

We need to introduce or recall some notation and conventions. Let a system \mathcal{R} involve contexts c_1, \dots, c_n ($n > 1$) and contents q_1, \dots, q_m ($m > 1$).

1. Notation related to a c-c system \mathcal{R} :

$$\mathcal{R} = \begin{pmatrix} \ddots & \vdots & \ddots & \\ \cdots & R_j^i & \cdots & (\text{bunch } R^i) \\ \ddots & \vdots & \ddots & \\ & \begin{pmatrix} \text{connection} \\ \mathcal{R}_j \end{pmatrix} & & \end{pmatrix}. \quad (38)$$

2. Corresponding notation for a (maximally connected) coupling S of \mathcal{R} :

$$S = \begin{pmatrix} \ddots & \vdots & \ddots & \\ \cdots & S_j^i & \cdots & \begin{pmatrix} \text{subcoupling} \\ S^i \end{pmatrix} \\ \ddots & \vdots & \ddots & \\ & \begin{pmatrix} \text{subcoupling} \\ S_j \end{pmatrix} & & \end{pmatrix}. \quad (39)$$

3. Notation for any (maximal) coupling T_j of a connection \mathcal{R}_j taken separately:

$$T_j = \begin{pmatrix} \vdots \\ T_j^i \\ \vdots \end{pmatrix}. \quad (40)$$

4. A value of a random variable R_j^i (hence also of S_j^i or T_j^i) is denoted v_j^i or w_j^i . A value of a bunch R^i (hence also of the subcoupling S^i of S) is denoted

v^i or w^i . We use v_j or w_j to denote values of couplings T_j and the corresponding subcouplings S_j of S (assumed to be maximally connected). The value v of S has the structure

$$v = \begin{pmatrix} \ddots & \vdots & \ddots & \\ \cdots & v_j^i & \cdots & \begin{pmatrix} \text{bunch} \\ \text{value } v^i \end{pmatrix} \\ \ddots & \vdots & \ddots & \\ & \begin{pmatrix} \text{connection} \\ \text{value } v_j \end{pmatrix} & & \end{pmatrix}. \quad (41)$$

As is customary, we use v, v^i, v_j, v_j^i sometimes as variables and sometimes as specific values of these variables.

5. Recall that in these matrices and vectors some entries are not defined: not every content is paired with every context. If q_j is measured (responded to) in context c_i , the random variable R_j^i exists, and the elements of the set V_j of its possible values can be enumerated $1, \dots, k_j$. Denoting

$$k = \max_{j=1, \dots, m} k_j, \quad (42)$$

without loss of generality, we can assume that

$$V_j = \{1, \dots, k\}, \quad (43)$$

for every $j = 1, \dots, m$. Indeed, one can always add values to V_j that occur with probability zero. The set of all values v of a coupling S is therefore $\{1, \dots, k\}^N$, where N is the number of all random variables in S .

6. We will refer to the values v of S as *hidden outcomes*. The term derives from quantum mechanics, where the problem of contextuality was initially presented as that of hidden variables (see the last paragraph of Section 2.2).

4.2. Linear equations associated with a c-c system

To specify a distribution of S , each of the hidden outcomes v should be assigned a probability mass $\gamma(v)$. Let us form a column vector \mathbf{Q} by arranging these $\gamma(v)$ -values in some, say, lexicographic order of v . Let us also form a column vector \mathbf{P} with the following structure:

$$\mathbf{P} = \begin{pmatrix} \text{bunch} & \text{connection} \\ \text{probabilities} & \text{probabilities} \\ \text{for } c_1, \dots, c_n & \text{for } q_1, \dots, q_m \end{pmatrix}. \quad (44)$$

Here,

$$\begin{aligned} \text{bunch} \\ \text{probabilities} &= (\Pr[R^i = v^i] : v^i \in \{1, \dots, k\}^{n_i}), \\ &\text{for } c_i \end{aligned} \quad (45)$$

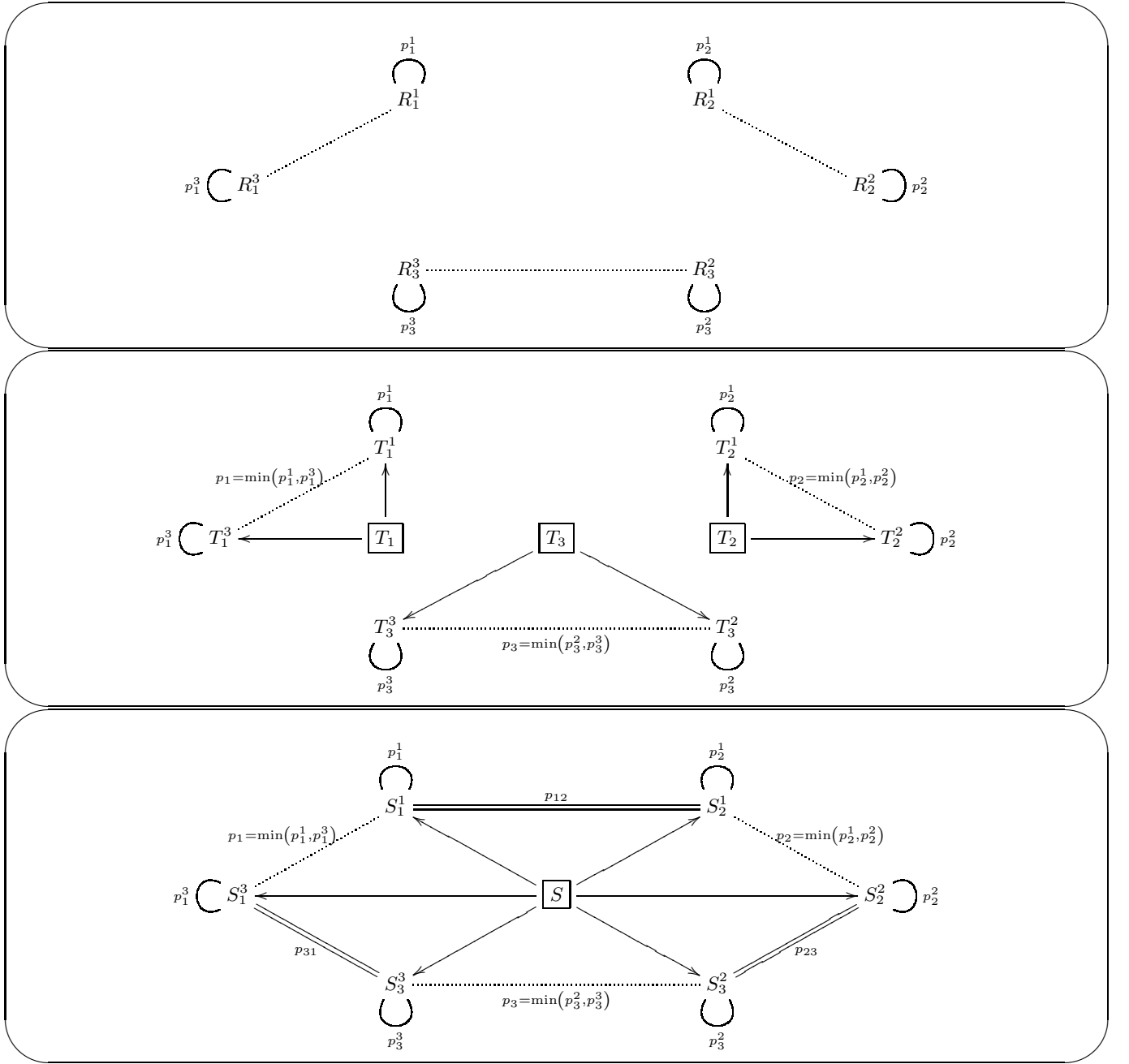


Figure 11. Each of the three connections of the SZLG system of Fig. 10, taken separately (top), can be coupled by a maximal coupling (middle). A hypothetical maximally connected coupling of the SZLG system (bottom) is one in which the subcouplings corresponding to the connections of the system are their maximal couplings. If such a coupling can be constructed (equivalently, if the maximal couplings in the middle panel are compatible with the system's bunches), then the system is noncontextual. It is possible that such a coupling does not exist, in which case the system is contextual.

where n_i is the number of elements in v^i . That is, the bunch probabilities for c_i are the joint probabilities that determine the distribution of the bunch R^i . The connection probabilities for q_j are the probabilities imposed by the maximal coupling T_j of the connection \mathcal{R}_j taken

separately:

$$\begin{aligned} \text{connection} \\ \text{probabilities} &= (\Pr [T_j = (l, \dots, l)] : l \in \{1, \dots, k\}) . \\ &\text{for } q_j \end{aligned} \quad (46)$$

Since S is a coupling of \mathcal{R} , we should have, for every

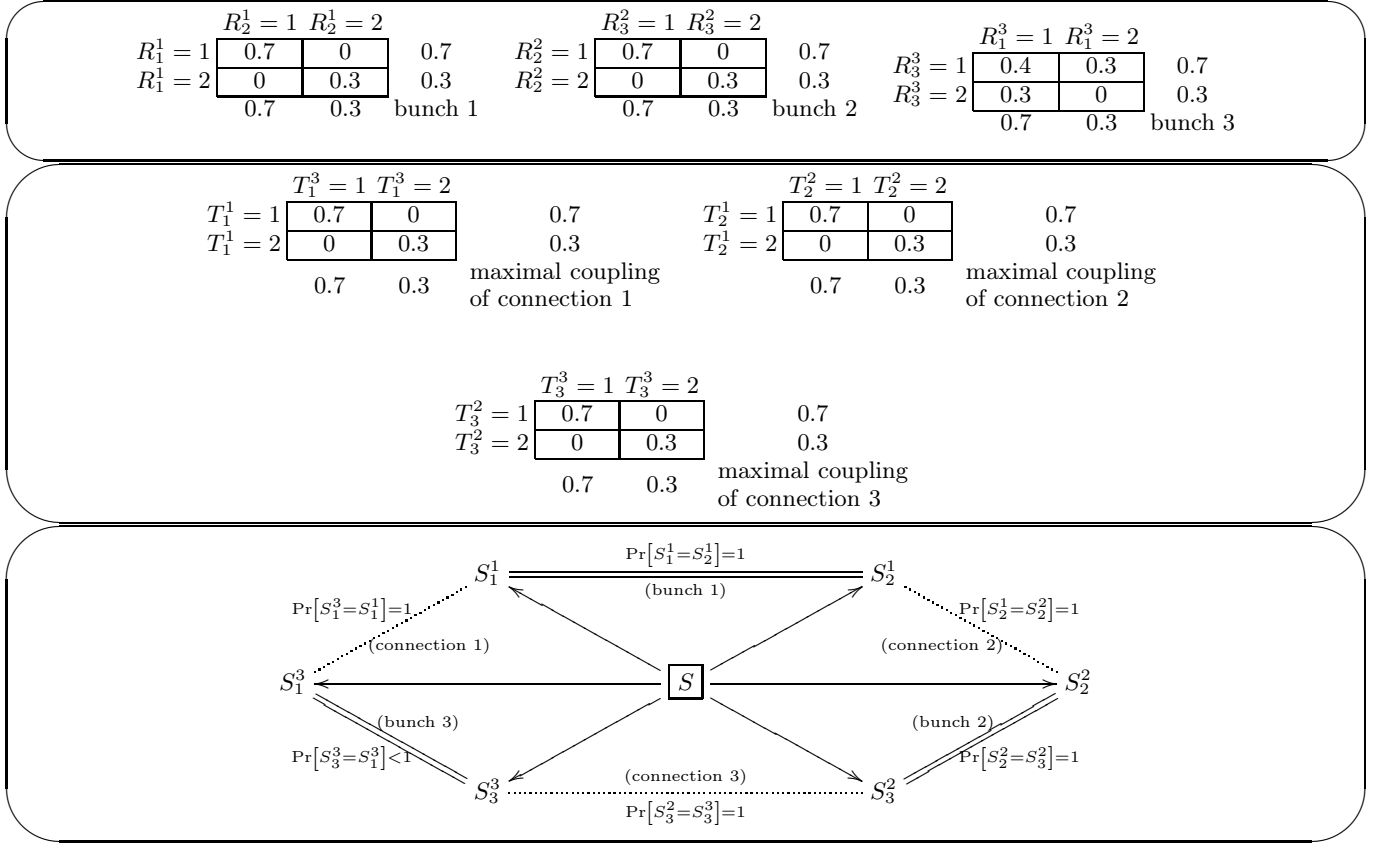


Figure 12. An example of a contextual SZLG system. The bunches are shown in the top panel. The middle panel shows the computed maximal couplings for the connections. The bottom panel shows that a maximally connected coupling does not exist because it is internally contradictory. To each edge of the diagram in this panel we append the probability with which the flanking random variables are equal to each other, and we show from where this probability is derived. Thus, $\Pr[S_1^1 = S_1^2] = 1$ is derived from the distribution of the first bunch, while $\Pr[S_1^3 = S_1^1] = 1$ is derived from the distribution of the maximal coupling of the first connection.

value w^i of every bunch R^i ,

$$\sum_v \lambda^i(v, w^i) \gamma(v) = \Pr[R^i = w^i], \quad (47)$$

where $\lambda^i(v, w^i) = 1$ if $v^i = w^i$ (i.e., if the i th row of v , in reference to (41), equals w^i), and $\lambda^i(v, w^i) = 0$ otherwise. Since S is a maximally connected coupling of \mathcal{R} , we should have, for every value $w_j = (l, \dots, l)$ of every maximal coupling T_j ,

$$\sum_v \lambda_j(v, w_j) \gamma(v) = \Pr[T_j = w_j = (l, \dots, l)] \quad (48)$$

where $\lambda_j(v, w_j) = 1$ if the j th column v_j of v in (41) equals w_j , and $\lambda_j(v, w_j) = 0$ otherwise.

In we list the hidden outcomes v in the same order as in the vector \mathbf{Q} , the 1/0 values of $\lambda^i(v, w^i)$ and 1/0 values of $\lambda_j(v, w_j)$ in (47) and (48) form rows of a Boolean matrix \mathbf{M} , one row per each (i, w^i) and each (j, w_j) , such that (47) and (48) can be written together as

$$\mathbf{M}\mathbf{Q} = \mathbf{P}. \quad (49)$$

We will refer to this matrix equation as the *system of equations associated with the c-c system \mathcal{R}* . In Section 6 below we will show that a vector of real numbers \mathbf{Q} satisfying this equation always exist. To form a distribution for a maximally connected coupling S , however, \mathbf{Q} also has to satisfy the following two constraints:

- (a): all components of \mathbf{Q} are nonnegative, and
- (b): they sum to 1.

The latter requirement is satisfied “automatically.” Indeed, by construction, the rows of \mathbf{M} corresponding to all possible values of any given bunch have pairwise disjoint cells containing 1’s: a hidden outcome v in (41) contains in its i th row one and only one value of the i th bunch. This means that if one adds all the rows of \mathbf{M} corresponding to the i th bunch one will get a row with 1’s in all cells. The scalar product of this row and \mathbf{Q} equals both the sum of the elements in \mathbf{Q} and the sum of all bunch probabilities in the i th bunch, which is 1.

The nonnegativity constraint, however, does not have to be satisfied: it is possible that every one of the infinite set of solutions for \mathbf{Q} contains some negative components.

hidden outcome, $\begin{pmatrix} v_1^1 & v_2^1 \\ v_1^2 & v_2^2 \end{pmatrix}$															
++	++	++	++	+-	+-	+-	+-	-+	-+	-+	-+	--	--	--	--
++	+-	-+	--	++	+-	-+	--	++	+-	-+	--	++	+-	-+	--
1	1	1	1												
				1	1	1	1								
								1	1	1	1				
												1	1	1	1
1				1				1				1			
	1				1				1				1		
		1				1				1				1	
			1				1				1				1
1	1			1	1										
										1	1			1	1
1		1						1		1					
					1		1					1		1	

$\Pr[S^1 = (+, +)]$ $= \Pr[R^1 = (+, +)]$
$\Pr[S^1 = (+, -)]$ $= \Pr[R^1 = (+, -)]$
$\Pr[S^1 = (-, +)]$ $= \Pr[R^1 = (-, +)]$
$\Pr[S^1 = (-, -)]$ $= \Pr[R^1 = (-, -)]$
$\Pr[S^2 = (+, +)]$ $= \Pr[R^2 = (+, +)]$
$\Pr[S^2 = (+, -)]$ $= \Pr[R^2 = (+, -)]$
$\Pr[S^2 = (-, +)]$ $= \Pr[R^2 = (-, +)]$
$\Pr[S^2 = (-, -)]$ $= \Pr[R^2 = (-, -)]$
$\Pr[S_1^1 = (+, +)]$ $= \Pr[T_1 = (+, +)]$
$\Pr[S_1^1 = (-, -)]$ $= \Pr[T_1 = (-, -)]$
$\Pr[S_2^1 = (+, +)]$ $= \Pr[T_2 = (+, +)]$
$\Pr[S_2^1 = (-, -)]$ $= \Pr[T_2 = (-, -)]$

Figure 13. The Boolean matrix \mathbf{M} (left) and vector \mathbf{P} (right) for the c-c system \mathcal{A} in Fig. 1. The values of the variables are encoded by ± 1 , with $+1$ shown by plus sign and -1 shown by minus sign. The first 8 elements of \mathbf{P} are bunch probabilities, the last 4 elements are connection probabilities.

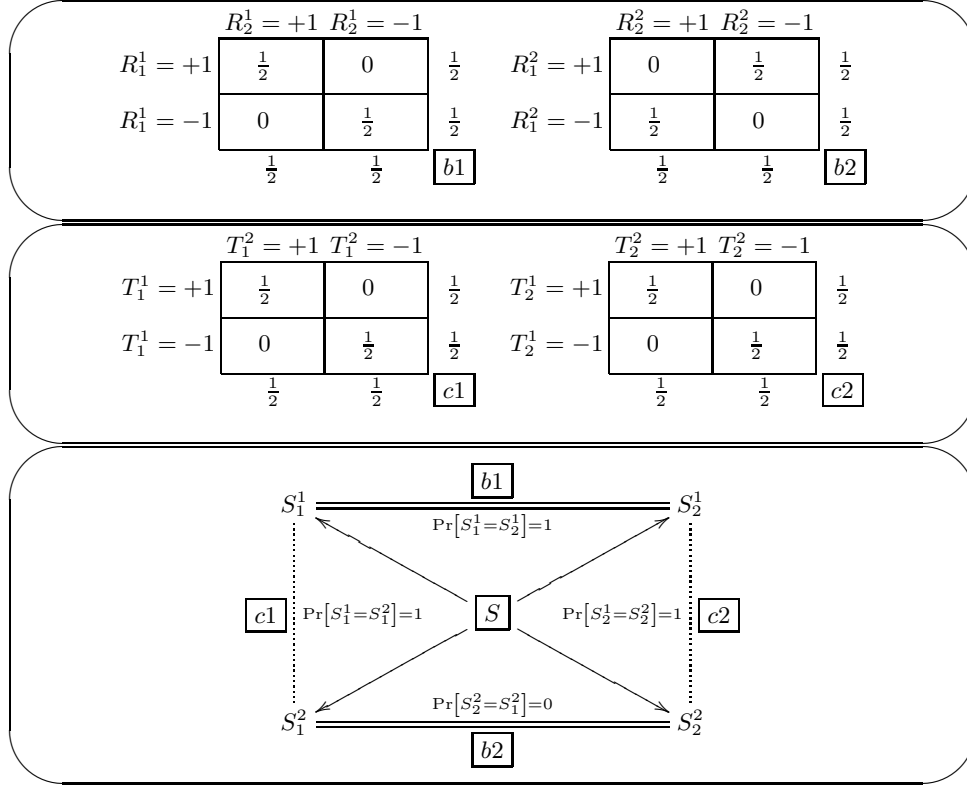


Figure 14. An example of a contextual c-c system of the \mathcal{A} -type (Fig. 1). The bunch probabilities are shown in the top panel. The middle panel shows the computed maximal connection probabilities. The bottom panel shows (using the same format and logic as in Fig. 12) that a maximally connected coupling S would be internally contradictory.

This is the case when the c-c system for which we have constructed the equations is contextual.

We can formulate now the main statement of this section.

Theorem 4.1. *A c-c system is noncontextual (i.e., it has a maximally connected coupling) if and only if the associated system of equations $\mathbf{MQ} = \mathbf{P}$ has a solution for \mathbf{Q} with nonnegative components. Any such a solution defines a distribution of the hidden outcomes of the coupling.*

The task of finding solutions for (49) subject to the nonnegativity constraint is a linear programming task. It is always well-defined and leads to an answer (an example of a solution or the determination that it does not exist) in polynomial time with respect to the number of the elements in \mathbf{Q} (Karmarkar, 1984). This is all that matters to us theoretically. In practice, some algorithms are more efficient than Karmarkar’s in most cases (e.g., the simplex algorithm).

The linear programming problem in Theorem 4.1 is especially transparent when all variables in a c-c system are binary with the same possible values, say, 1 and -1. The reader may find it useful to check, using Fig. 13, all the steps of the derivation of the linear equations (49) using the c-c system \mathcal{A} of our opening example (Fig. 1). Whether this system is contextual depends on \mathbf{P} , specifically, on the bunch probabilities in \mathbf{P} . Recall that the connection probabilities, the last four elements of \mathbf{P} in Fig. 13, are computed from the bunch probabilities using Theorem 3.3. Thus, if the bunch probabilities in \mathbf{P} are as shown in the upper panel of Fig. 14, then the connection probabilities should be as in the middle panel, and it can be shown by applying a linear programming algorithm that the matrix equation $\mathbf{MQ} = \mathbf{P}$ does not have a solution with nonnegative elements. In this simple case we can confirm this result by a direct observation of the internal contradiction in the maximally connected coupling shown in the bottom panel.

5. CYCLIC C-C SYSTEMS

The question we pose now is: is there a shortcut to find out if a c-c system is contextual, without resorting to linear programming? As it turns out, for an important class of so-called cyclic systems with binary variables (Dzhafarov, Zhang, & Kujala, 2015; Dzhafarov, Kujala, & Cervantes, 2016; Kujala, Dzhafarov, & Larsson, 2015; Kujala & Dzhafarov, 2016) the answer to this question is affirmative.

5.1. Contextuality criterion for cyclic c-c systems

A *cyclic* system is defined as a c-c system in which

(CYC1): each conteXt includes precisely two conteNts,

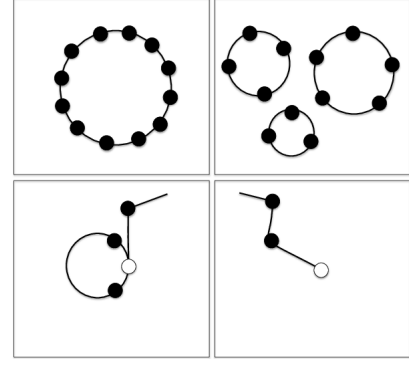


Figure 15. The conteNts (small circles) in a system satisfying the conditions CYC1-CYC2 can be arranged in a cycle (upper left panel) or several disjoint cycles (upper right panel), such that any two adjacent conteNts define one conteXt. The proof is in the lower panels: for the conteNts not to form cycles, some conteNt should be placed in a position like the ones of the open circles in the two lower panels. This is, however, impossible: in the left one it belongs to more than two conteXts (identified with a pair of conteNts), in the right one it belongs to a single conteXt.

(CYC2): each conteNt is included in precisely two conteXts.

We will also assume that

(CYC3): all random variables are binary with the same two possible values (traditionally, 1 and -1).

Fig. 15 makes it clear why such a system is called cyclic: to satisfy the properties above, the conteNts should be arrangeable in one or more cycles in which a conteXt corresponds to any two adjacent conteNts. If the conteNts are arranged into several cycles, from the point of view of contextuality analysis each cycle forms a separate system, with no information regarding one of them being relevant for another’s analysis. We will therefore, with no loss of generality, assume that a cyclic system involves a single cycle.

The number of conteNts (or connections) in a cyclic system equals the number of conteXts (or bunches) in it, and it is referred to as the *rank* of the system. The c-c matrix for the cyclic system has the form shown in Fig. 16, generalizing the matrices in Fig. 1 (cyclic system of rank 2) and Fig. 9 (cyclic system of rank 3).

In the presentation below we use $\langle X \rangle$ to denote the expected value of a random variable X with possible values +1 and -1:

$$\langle X \rangle = \Pr[X = 1] - \Pr[X = -1]. \quad (50)$$

Given $k > 0$ real numbers x_1, \dots, x_k , we define the function

$$\text{Sodd}(x_1, \dots, x_k) = \max_{(\iota_1, \dots, \iota_k) \in \{-1, 1\}^k: \prod_{i=1}^k \iota_i = -1} \left(\sum_{i=1}^k \iota_i x_i \right). \quad (51)$$

R_1^1	R_2^1	\cdot	\cdot	\cdots	\cdot	\cdot	c_1
\cdot	R_2^2	R_3^2	\cdot	\cdots	\cdot	\cdot	c_2
\cdot	\cdot	R_3^3	R_4^3	\cdots	\cdot	\cdot	c_3
\vdots	\vdots	\vdots	\vdots	\ddots	\vdots	\vdots	\vdots
\cdot	\cdot	\cdot	\cdot	\cdots	R_{n-1}^{n-1}	R_n^{n-1}	c_{n-1}
R_1^n	\cdot	\cdot	\cdot	\cdots	\cdot	R_n^n	c_n
q_1	q_2	q_3	q_4	\cdots	q_{n-1}	q_n	<div style="border: 1px solid black; padding: 2px; display: inline-block;">CYC</div>

Figure 16. The c-c matrix for a cyclic system of an arbitrary rank n (shown here for a sufficiently large n , but n can be as small as 2 or 3, shown in Figs. 1 and 9, respectively). Each conteXt includes two conteNts, and each conteNt is included in two conteXts. Bunches of the system are formed by the pairs of random variables $(R_1^1, R_2^1), (R_2^2, R_3^2), \dots, (R_{n-1}^{n-1}, R_n^{n-1}), (R_n^n, R_1^n)$, whereas the connections of the system are formed by the pairs $(R_1^1, R_1^n), (R_2^2, R_2^n), \dots, (R_{n-1}^{n-1}, R_{n-1}^n), (R_n^n, R_n^n)$. All random variables are binary, with possible values denoted +1 and -1.

This means that one takes each argument x_i in the sum with either + sign or - sign, tries all combinations in which the number of minuses is odd, and chooses the largest sum. For example,

$$s_{\text{odd}}(5, 6) = -5 + 6,$$

$$s_{\text{odd}}(5, -6) = 5 - (-6),$$

$$s_{\text{odd}}(1, 2, -3, -10, 100) = -1 + 2 - (-3) - (-10) + 100.$$

Finally, we have to introduce the cyclic addition and subtraction operations, $\oplus 1$ and $\ominus 1$: if the numbers $1, \dots, n$ are arranged circularly like on a clock dial, then $\oplus 1$ and $\ominus 1$ mean, respectively, clockwise and counterclockwise shift to the next position. The only difference of these operations from the usual +1 and -1 is that $n \oplus 1 = 1$ and $1 \ominus 1 = n$.

Now we can formulate a criterion of (i.e., a necessary and sufficient condition for) contextuality of a cyclic system.

Theorem 5.1 (Kujala and Dzhaferov, 2016). *A cyclic system of rank n is noncontextual if and only if*

$$s_{\text{odd}}(\langle R_i^i R_{i \oplus 1}^i \rangle : i = 1, \dots, n) \leq n-2 + \sum_{i=1}^n |\langle R_i^i \rangle - \langle R_i^{i \ominus 1} \rangle|. \quad (52)$$

In the left-hand side expression, the arguments of the function s_{odd} are the expected products for the n bunches of the system: $\langle R_1^1 R_2^1 \rangle, \langle R_2^2 R_3^2 \rangle$, etc., the last one, due to the cyclicity, being $\langle R_n^n R_1^n \rangle$. In the right-hand side of the inequality, the summation sign operates

over the n connections of the system: for each connection, $(R_1^1, R_1^n), (R_2^2, R_2^n), \dots, (R_n^n, R_n^{n-1})$, we take the distance between the expectations of its elements. If the system is consistently connected, all these distances are zero, and the criterion acquires the form

$$s_{\text{odd}}(\langle R_i^i R_{i \oplus 1}^i \rangle : i = 1, \dots, n) \leq n-2. \quad (53)$$

5.2. Examples of cyclic systems

It has been mentioned in the introduction that cyclic systems of rank 2 have been prominently studied in a behavioral setting, in the paradigm where the conteNts are two Yes/No questions and conteXts are defined by two orders in which these questions are asked. The noncontextuality criterion (52) for $n = 2$ acquires the form

$$|\langle R_1^1 R_2^1 \rangle - \langle R_2^2 R_1^2 \rangle| \leq |\langle R_1^1 \rangle - \langle R_1^2 \rangle| + |\langle R_2^1 \rangle - \langle R_2^2 \rangle|. \quad (54)$$

It is known (Moore, 2002) that the distributions of responses to the same question depend on whether the question is asked first or second. In our terminology, this means that the system is inconsistently connected, and the right-hand side of the inequality above is greater than zero. At the same time, as Wang and Busemeyer (2013) have discovered in their analysis of a large body of question pairs, the probability with which the answer to the two questions is one and the same does not depend on the order in which they are asked. To the extent this generalization holds, it means that the left-hand side of the inequality (54) is zero. In turn, this means that the system describing responses to two questions asked in two orders cannot be contextual (see Dzhaferov, Zhang, Kujala, 2015, for a detailed discussion).

Perhaps the best known cyclic system is one of rank 4, whose quantum-mechanical version is shown in Fig. 17. According to the laws of quantum mechanics, the product expectation $\langle R_i^i R_{i \oplus 1}^i \rangle$ for Alice's choice of axis q_i and Bob's choice of axis $q_{i \oplus 1}$ equals $-\cos \theta_{i, i \oplus 1}$, where $\theta_{i, i \oplus 1}$ denotes the angle between the two axes. Assume, e.g., that the four axes are coplanar, and form the following angles with respect to some fixed direction

$$\begin{matrix} q_1 & q_2 & q_3 & q_4 \\ 0 & \pi/4 & \pi/2 & -\pi/4 \end{matrix}. \quad (55)$$

The calculation yields in this case

$$s_{\text{odd}}(\langle R_i^i R_{i \oplus 1}^i \rangle : i = 1, 2, 3, 4) = 2\sqrt{2}. \quad (56)$$

If any possibility of direct interaction between Alice and Bob is excluded, i.e., Alice's measurements are not influenced by Bob's choices of his axes and Bob's measurements are not influenced by Alice's choices of her axes, and if we exclude any possibility of misrecording, then

$$\sum_{i=1}^4 |\langle R_i^i \rangle - \langle R_i^{i \ominus 1} \rangle| = 0, \quad (57)$$

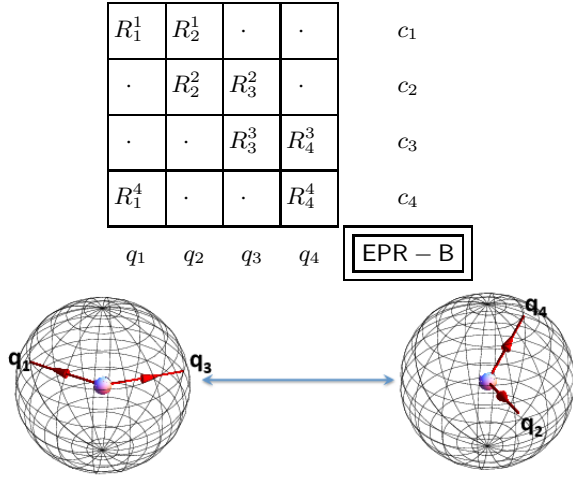


Figure 17. Einstein-Podolsky-Rosen-Bohm-system (Bohm & Aharonov, 1957), for which the celebrated Bell-CHSH inequalities were derived (Bell, 1964, 1966; Clauser et al., 1969; Clauser & Horne, 1974). This is a cyclic system of rank 4. Two spin-1/2 particles (e.g., electrons) are created in what is called a “singlet state” and move away from each other while remaining entangled. Alice chooses one of the two axes denoted 1 and 3 and measures the spin of the left particle, “up” (+1) or “down” (−1). Bob chooses one of the two axes denoted 2 and 4 and measures the spin of the right particle, +1 or −1. The conteNts here are the (choices of the) four axes, q_1, q_2, q_3, q_4 , the conteXts are defined by the pairs of the axes chosen, one by Alice and another by Bob. The system is contextual for some combinations of the four axes.

and the inequality (52) acquires the form of the inequality (53), for $n = 4$. The value $2\sqrt{2}$ for the left-hand side expression violates this inequality, indicating that the system is contextual.

There were several studies of systems having the cyclic rank 4 structure in behavioral settings. Thus, Fig. 18 describes one of the psychophysical matching experiments analyzed in Dzhamfarov, Ru, and Kujala (2015). The dichotomization of the response variables was done as follows: we choose radial length values rad_1, rad_3 (they may be the same) and polar angle values ang_2 and ang_4 (they also may be the same), and we define

$$R_i^i = \begin{cases} +1 & \text{if } Rad_{i,i\oplus 1} > rad_i \\ -1 & \text{if } Rad_{i,i\oplus 1} \leq rad_i \end{cases}, \quad (58)$$

$$R_{i\oplus 1}^i = \begin{cases} +1 & \text{if } Ang_{i,i\oplus 1} > ang_{i\oplus 1} \\ -1 & \text{if } Ang_{i,i\oplus 1} \leq ang_{i\oplus 1} \end{cases}$$

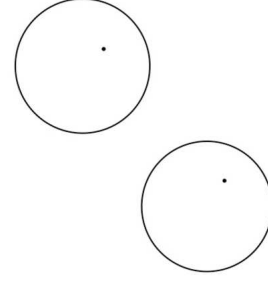


Figure 18. A matching experiment: a participant rotated a trackball that controlled the position of the dot within a lower-right circle until she judged it to match the fixed position of the target dot in the upper-left circle. The positions were described in polar coordinates, and the target position could have one of two radius values q_1 or q_3 combined with one of two polar angles, q_2 or q_4 . In each of the four conteXts, $(q_i, q_{i\oplus 1})$, $i = 1, 2, 3, 4$, the adjusted dot’s position was described by two polar coordinates (random variables) $Rad_{i,i\oplus 1}$ and $Ang_{i,i\oplus 1}$, that were then dichotomized to create a cyclic system of rank 4.

for $i = 1, 3$, and

$$R_i^i = \begin{cases} +1 & \text{if } Ang_{i,i\oplus 1} > ang_i \\ -1 & \text{if } Ang_{i,i\oplus 1} \leq ang_i \end{cases}, \quad (59)$$

$$R_{i\oplus 1}^i = \begin{cases} +1 & \text{if } Rad_{i,i\oplus 1} > rad_{i\oplus 1} \\ -1 & \text{if } Rad_{i,i\oplus 1} \leq rad_{i\oplus 1} \end{cases}$$

for $i = 2, 4$. The parameters rad_1, rad_3 and ang_2, ang_4 can be chosen in multiple ways, paralleling various sets of four axes in the Alice-Bob quantum-mechanical experiment.

See Dzhamfarov, Ru, and Kujala (2015) for other examples of behavioral rank 4 cyclic systems. That paper also reviews a behavioral experiment with a cyclic system of rank 3. Cyclic systems of rank 5 play an important role in quantum theory (Klyachko et al., 2008). For the contextuality analysis of an experiment designed to test (a special form of) the inequality (52) for $n = 5$ (Lapkiewicz et al., 2011), see Kujala, Dzhamfarov, and Larsson (2015).

6. HOW TO MEASURE DEGREE OF CONTEXTUALITY

Intuitively, some contextual systems are more contextual than others. For instance, a cyclic system of rank n can violate the inequality (52) “grossly” or “slightly,” and in the latter case it may be considered less contextual. The question we pose now is: if a c-c system is contextual, is there a *principled* way to measure the degree of contextuality in it? The emphasis is on the qualifier “principled,” as one can easily come up with various ad hoc measures for special types of c-c systems.

In this section we describe one way of constructing such a measure. It uses the notion of *quasi-probability distributions* that differ from the proper ones in that the probability masses in them are replaced with arbitrary, possibly negative, real numbers summing to unity. This conceptual tool has been previously used to deal with contextuality in consistently connected systems (Abramsky & Brandenburger, 2011; Al-Safi & Short, 2013). A measure of contextuality based on the notion of quasi-probability distributions, also for consistently connected systems, was proposed by de Barros and Oas (2014) and investigated in de Barros, Oas, and Suppes (2015) and de Barros et al. (2015). Our measure is a generalization of the de Barros-Oas measure to arbitrary c-c systems. Another generalization and a different way of using quasi-probability distributions to measure contextuality in arbitrary c-c systems was recently proposed by Kujala (2016). This measure requires a modification of CbD and will not be discussed here.

6.1. Dropping the nonnegativity constraint

We have seen that a c-c system is contextual if and only if the associated system of linear equations $\mathbf{M}\mathbf{Q} = \mathbf{P}$ does not have a solution for \mathbf{Q} with nonnegative components. In this section we show that if we drop the nonnegativity constraint the system of linear equations always has a solution (and generally an infinity of them). Any such a solution assigns real numbers to all hidden outcomes of the hypothetical maximally connected coupling. Some of these numbers can be negative and some may exceed 1, but they sum to 1.

The existence of \mathbf{Q} solving the linear equations $\mathbf{M}\mathbf{Q} = \mathbf{P}$ follows from the existence of \mathbf{Q} solving another system of linear equations,

$$\mathbf{M}^*\mathbf{Q} = \mathbf{P}^*. \quad (60)$$

We will refer to it as a *modified-and-expanded* system of linear equations associated with a c-c system. The term reflects the fact that in \mathbf{P}^* as compared to \mathbf{P} the bunch probabilities are presented in a modified form, and the connection probabilities are expanded to specify entire distributions of the (maximal) couplings of all connections. The rows of the Boolean matrix \mathbf{M}^* as compared to \mathbf{M} change accordingly, although its columns remain corresponding to the hidden outcomes v ordered in the same way as $\gamma(v)$ are ordered in \mathbf{Q} .

The construction of \mathbf{P}^* and \mathbf{M}^* consists of three parts.

Part 1: first row. The first element of \mathbf{P}^* is 1, and the first row of \mathbf{M}^* is filled with 1's. This choice ensures

$$\sum_v \gamma(v) = 1. \quad (61)$$

Part 2: bunch probabilities. Next we include in \mathbf{P}^* the 1-marginal probabilities $\Pr[R_j^i = l]$ for all random variables R_j^i and all $l = 1, \dots, k-1$. The value $l = k$ is

excluded because $\Pr[R_j^i = k]$ is uniquely determined as a linear combination of the probabilities already included. The row of \mathbf{M}^* corresponding to $\Pr[R_j^i = l]$ (i.e., the row whose scalar product by \mathbf{Q} yields this probability) has 1's in the cells for v with $v_j^i = l$, and it has zeros in other cells.

The next set of elements of \mathbf{P}^* are 2-marginal probabilities $\Pr[R_j^i = l, R_{j'}^{i'} = l']$ for all pairs of random variables $R_j^i, R_{j'}^{i'}$ ($j < j'$) and $(l, l') \in \{1, \dots, k-1\}^2$. The 2-marginal probabilities for $R_j^i = k$ or $R_{j'}^{i'} = k$ are excluded because they are uniquely determinable as linear combinations of the probabilities already included. The row of \mathbf{M}^* corresponding to $\Pr[R_j^i = l, R_{j'}^{i'} = l']$ has 1's in the cells for v with $v_j^i = l, v_{j'}^{i'} = l'$, and it has zeros in other cells.

Proceeding in this manner, we include in \mathbf{P}^* the r -marginal probabilities $\Pr[R_{j_1}^{i_1} = l_1, \dots, R_{j_r}^{i_r} = l_r]$ ($j_1 < \dots < j_r$) for all bunches R^i with at least r distinct random variables, and for all $(l_1, \dots, l_r) \in \{1, \dots, k-1\}^r$. We exclude all probabilities involving the value k for at least one of the random variables in the r -marginal. The row in \mathbf{M}^* corresponding to $\Pr[R_{j_1}^{i_1} = l_1, \dots, R_{j_r}^{i_r} = l_r]$ has 1's in the cells for v with $v_{j_1}^{i_1} = l_1, \dots, v_{j_r}^{i_r} = l_r$, and it has zeros in other cells. The procedure stops at the smallest r such that no bunches in the system contain r distinct random variables.

Part 3: connection probabilities. This part of the construction deals with the maximal couplings for connections. By Theorem 3.3, a maximal coupling T_j for a connection \mathcal{R}_j always exists, i.e., one can always find the joint probabilities $\Pr[T_j = v_j]$ for all v_j , so that

$$\Pr[T_j^i = v_j^i] = \Pr[R_j^i = v_j^i] \quad (62)$$

for all R_j^i in \mathcal{R}_j , and

$$\Pr[T_j = (l, \dots, l)] = \min_{\text{components } T_j^i \text{ of } T_j} \Pr[T_j^i = l] \quad (63)$$

for $l = 1, \dots, k$. We choose any maximal coupling T_j for each connection \mathcal{R}_j , and we treat it as if it were an observed bunch. This allows us to repeat on T_j the procedure of Part 2, except that the 1-marginal probabilities $\Pr[T_j^i = v_j^i] = \Pr[R_j^i = l]$, which are the same for both bunches and connections, have already been included in \mathbf{P}^* (and the corresponding rows in \mathbf{M}^* have been formed). We add, however, all higher-order marginals $\Pr[T_j^{i_1} = l_1, \dots, T_j^{i_r} = l_r]$ ($i_1 < \dots < i_r$) for all connections \mathcal{R}_j with at least r distinct random variables, for all $(l_1, \dots, l_r) \in \{1, \dots, k-1\}^r$. The row in \mathbf{M}^* corresponding to $\Pr[T_j^{i_1} = l_1, \dots, T_j^{i_r} = l_r]$ has 1's in the cells for v with $v_{j_1}^{i_1} = l_1, \dots, v_{j_r}^{i_r} = l_r$, and it has zeros in other cells. We apply this procedure to $r = 2, 3, \dots$ until we reach the smallest r such that no connection in the system contains r distinct random variables.

This completes the construction of \mathbf{P}^* and \mathbf{M}^* . As an example, Fig. 19 shows \mathbf{P}^* and \mathbf{M}^* for the system \mathcal{A} of Fig. 1, whose associated \mathbf{P} and \mathbf{M} are shown in

\mathbf{M}^*																\mathbf{P}^*
++	++	++	++	+-	+-	+-	+-	-+	-+	-+	-+	-+	-+	-+	-+	
1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	$\Pr[\] = 1$
1	1	1	1	1	1	1	1									$\Pr[S_1^1 = +]$ $= \Pr[R_1^1 = +]$
1	1			1	1			1	1				1	1		$\Pr[S_2^1 = +]$ $= \Pr[R_2^1 = +]$
1		1		1		1		1		1		1		1		$\Pr[S_1^2 = +]$ $= \Pr[R_1^2 = +]$
1		1		1		1		1		1		1		1		$\Pr[S_2^2 = +]$ $= \Pr[R_2^2 = +]$
1	1	1	1													$\Pr[S_1^1 = +, S_2^1 = +]$ $= \Pr[R_1^1 = +, R_2^1 = +]$
1				1				1					1			$\Pr[S_1^2 = +, S_2^2 = +]$ $= \Pr[R_1^2 = +, R_2^2 = +]$
1	1			1	1											$\Pr[S_1^1 = +, S_1^2 = +]$ $= \Pr[T_1^1 = +, T_1^2 = +]$
1		1						1		1						$\Pr[S_2^1 = +, S_2^2 = +]$ $= \Pr[T_2^1 = +, T_2^2 = +]$

Figure 19. The Boolean matrix \mathbf{M}^* and vector \mathbf{P}^* for the c-c system \mathcal{A} in Fig. 1 (cyclic system of rank 2). The first row of \mathbf{M}^* , formally, corresponds to the “0-marginal” probability in \mathbf{P} , and its role is to ensure that the elements of \mathbf{Q} in $\mathbf{M}^* \mathbf{Q} = \mathbf{P}^*$ sum to 1. The next 6 rows are modified bunch probabilities and the last 2 rows are connection probabilities (“expanded,” but in this case nominally only).

\mathbf{M}^*																\mathbf{P}^*
++	++	++	++	+-	+-	+-	+-	-+	-+	-+	-+	-+	-+	-+	-+	
1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	0
1	1	1	1	1	1	1	1									0
1	1			1	1			1	1				1	1		0
1		1		1		1		1		1		1		1		1/2
1		1		1		1		1		1		1		1		1/2
1	1	1	1													0
1	1	1	1													1/2
1				1				1					1			0
1	1			1	1											1/2
1		1						1		1						0
1								1								1/2
1								1								0
1		1						1		1						1/2

Figure 20. The same as Fig. 19, but with the numerical values of the bunch and connection probabilities of the contextual system shown in Fig. 14. The inserted column \mathbf{Q} of quasi-probabilities is a solution for $\mathbf{M}^* \mathbf{Q} = \mathbf{P}^*$.

Fig. 13. Since all connections in this system contain just two binary variables (as in any cyclic system), the expanded connection probabilities in this case are uniquely determined by the 1-marginal (bunch) probabilities and the joint probabilities with which the coupled variables attain the value 1. In more complex systems the expanded connection probabilities for maximal couplings can be specified in an infinity of ways.

It can now be shown that the rows in \mathbf{M}^* are linearly independent. Indeed, consider a linear combination of these rows that equals the null vector.

$$\alpha_1 (\text{row}_1) + \alpha_2 (\text{row}_2) + \dots + \alpha_{N_{\text{rows}}} (\text{row}_{N_{\text{rows}}}) = \mathbf{0}. \quad (64)$$

The first row of \mathbf{M}^* consists of 1’s only, and this includes the entry 1 in the column of \mathbf{M}^* corresponding to

the hidden outcome v with all elements in it equal to k . Any other row in \mathbf{M}^* contains zero in this column. Indeed, entry 1 would have meant that the corresponding probability in \mathbf{P}^* was computed for at least one random variable attaining a value belonging to v : $S_j^i = v_j^i = k$. But this is impossible because the value k is not used in any of the probabilities in \mathbf{P}^* . It follows that $\alpha_1 = 0$. Without loss of generality therefore, we can eliminate this row from consideration.

The row corresponding to a 1-marginal $\Pr[S_j^i = l]$ ($l < k$) contains 1 in the column corresponding to the hidden outcome v with $v_j^i = l$ and other entries equal to k . All other rows of \mathbf{M}^* , now that we have eliminated the first row, contain zero in the column corresponding to this v . Indeed, to have 1 for this v in another row

would have meant that the corresponding probability in \mathbf{P}^* was computed for the conjunction of $S_j^i = l$ with at least one other random variable attaining a value belonging to v : $S_{j'}^i = v_{j'}^i$ or $S_{j'}^{i'} = v_{j'}^{i'}$. But all other elements of v equal k , and the value k is not used in any of the probabilities in \mathbf{P}^* . It follows that the α -coefficients in (64) are zero for all rows corresponding to 1-marginal probabilities. Consequently we can consider all these rows eliminated.

The row corresponding to a 2-marginal $\Pr[S_j^i = l, S_{j'}^{i'} = l']$ (where $i = i'$ if this is a bunch probability, or $j = j'$ if it is a connection probability, $l, l' < k$) contains 1 in the column corresponding to the hidden outcome v with $v_j^i = l, v_{j'}^{i'} = l'$ and all other values equal to k . All other rows in this column, now that we have eliminated the first row and all 1-marginal rows, contain zero in the column corresponding to this v . Indeed, to have 1 for this v in another row would have meant that the corresponding probability in \mathbf{P}^* was computed for the conjunction of $S_j^i = l, S_{j'}^{i'} = l'$ with at least one other random variable attaining a value belonging to v : $S_{j''}^{i''} = v_{j''}^{i''}$ (where $i'' = i' = i$ or $j'' = j' = j$), which is impossible as all other elements of v equal k . It follows that the α -coefficients in (64) are zero for all rows corresponding to 2-marginal (bunch and connection) probabilities. Consequently we can consider all these rows eliminated.

Proceeding in this manner to higher-order marginals until all of them are exhausted, we prove that the rows in \mathbf{M}^* are linearly independent. It follows that the system of equations $\mathbf{M}^* \mathbf{Q} = \mathbf{P}^*$ always has solutions for \mathbf{Q} with real-number components (summing to 1). Since \mathbf{M} and \mathbf{P} in the original system of linear equations $\mathbf{M} \mathbf{Q} = \mathbf{P}$ associated with a given c-c system are obtained as one and the same linear combination of the rows of, respectively, \mathbf{M}^* and \mathbf{P}^* , any solution of $\mathbf{M}^* \mathbf{Q} = \mathbf{P}^*$ is also a solution of $\mathbf{M} \mathbf{Q} = \mathbf{P}$.

Theorem 6.1. *Any modified-and-expanded system of equations $\mathbf{M}^* \mathbf{Q} = \mathbf{P}^*$ associated with a c-c system has a solution for \mathbf{Q} whose components are real numbers summing to 1. Any such solution is also a solution for the original system of equations $\mathbf{M} \mathbf{Q} = \mathbf{P}$ associated with the same c-c system.*

In relation to the possible generalization of CbD discussed in the concluding section of this paper, note that nowhere in the proof of this theorem did we use the fact that a quasi-coupling S for the system \mathcal{R} is maximally connected. In other words, the proof and the construction of \mathbf{M}^* and \mathbf{P}^* make no use of the maximality constraint (63). The only fact that matters is that every connection taken separately is coupled in some way, so that all connection probabilities in \mathbf{P}^* are well-defined.

6.2. Quasi-probabilities and quasi-couplings

Let us call the components $\gamma(v)$ of \mathbf{Q} in Theorem 6.1 (*signed*) *quasi-probability masses*, and let us call the function γ (*signed*) *quasi-probability distribution*. Using this terminology, the system of linear equations $\mathbf{M}^* \mathbf{Q} = \mathbf{P}^*$ (hence also $\mathbf{M} \mathbf{Q} = \mathbf{P}$) always produces (generally an infinity of) quasi-probability distributions of hidden outcomes as solutions for \mathbf{Q} .

If the quasi-probability distribution of the hidden outcomes is not a proper probability distribution, then it does not define a coupling for the c-c system we are dealing with. However, we can introduce the notion of a (maximally connected) *quasi-coupling*, by replicating the definition of a (maximally connected) coupling, but with all references to probabilities being replaced with quasi-probabilities.

A *quasi-random variable* X in general is defined as a pair

$$X = (\text{id}X, \text{qdi}X), \quad (65)$$

where $\text{id}X$ is as before and $\text{qdi}X$ is a quasi-probability distribution function mapping a (finite) set V_X of possible values of X into the set of reals,

$$\text{qdi}X : V_X \rightarrow \mathbb{R}. \quad (66)$$

The only constraint is that

$$\sum_{v \in V_X} \text{qdi}X(v) = 1. \quad (67)$$

For any subset V of V_X we define quasi-probabilities⁸

$$\text{qPr}[X \in V] = \sum_{v \in V} \text{qdi}X(v).$$

The rest of the conceptual set-up (the class \mathbf{E} generated from a base set \mathbf{R} , the notion of jointly distributed quasi-random variables, their marginals, functions, etc.) precisely parallels one for the ordinary, or proper probability distributions and random variables. We can safely omit details.

Definition 6.2. A *quasi-coupling* $S_{\mathcal{R}}$ of a c-c system \mathcal{R} is defined as a set of jointly distributed quasi-random variables in a one-to-one correspondence with the union of the components of the bunches of \mathcal{R} , such that the quasi-probability distribution of every marginal of $S_{\mathcal{R}}$ that corresponds to a bunch of the system coincides with the (proper) distribution of this bunch. A quasi-coupling $S_{\mathcal{R}}$ of \mathcal{R} is *maximally connected* if every marginal of $S_{\mathcal{R}}$ that corresponds to a connection of the system is a maximal coupling for this connection.

⁸ This is a specialization of the measure-theoretic notion of *signed measure* (or *charge*) to probability spaces with finite sets.

Let us illustrate this definition on the contextual rank 2 cyclic system shown in Fig. 14. The set of hidden outcomes here consists of the 16 four-component combinations of 1's and -1's shown in Fig. 19. The contextuality of this system means that these hidden outcomes cannot be assigned proper probabilities. We can, however, assign real numbers to these outcomes as shown in Fig. 20.

1. Taking the scalar product of this vector of numbers with the first row (i.e., summing these numbers), we get 1. This shows that our assignment of the numbers is a quasi-probability distribution, so we can consider a quasi-random variable S with this distribution.
2. Taking the scalar product of the quasi-probability masses with the subsequent six rows, we get numerical values that are equal to the corresponding (proper) probabilities characterizing the bunches of the system in Fig. 14. This shows that S is a quasi-coupling of this system.
3. Finally, the scalar products of the quasi-probability masses with the last two rows yield the values of the probabilities characterizing the maximal couplings for the connections of the system in Fig. 14. This shows that S is a maximally connected quasi-coupling of this system.

6.3. Contextuality measure based on quasi-couplings

Proper probability distributions are quasi-probability distributions with no negative quasi-probability masses. If X is a quasi-random variable, the value

$$\|X\| = \sum_{v \in V_X} |\text{qdi}X(v)| \quad (68)$$

is known in the theory of signed measures as the *total variation* of $\text{qdi}X$ (or simply of X). Its value is 1 if X is a proper random variable. Otherwise $\|X\| > 1$, and the excess of $\|X\|$ over 1 can be thought of as a measure of “improperness” of X .

Consider the set of all maximally connected quasi-couplings $S_{\mathcal{R}}$ of a c-c system. We are now interested in the total variations $\|S_{\mathcal{R}}\|$ of (the distributions of) the quasi-couplings. The set of these values is bounded from below by 1, therefore it has an infimum,

$$t = \inf \|S_{\mathcal{R}}\|. \quad (69)$$

It can be readily seen that this infimum is in fact a minimum of all $\|S_{\mathcal{R}}\|$, i.e., that the set of all $S_{\mathcal{R}}$ contains a quasi-coupling $S_{\mathcal{R}}^*$ with

$$\|S_{\mathcal{R}}^*\| = t. \quad (70)$$

Indeed, if the set of all $S_{\mathcal{R}}$ is finite, the statement is trivially true; otherwise we choose an infinite sequence of

$\|S_{\mathcal{R}}\|$ converging to t . Without loss of generality, we can assume for all members of this sequence

$$\|S_{\mathcal{R}}\| - t < \varepsilon. \quad (71)$$

It follows that the quasi-probability masses $\text{qdi}S_{\mathcal{R}}(v)$ for all hidden outcomes v in all these quasi-couplings are confined within a closed interval $[-t - \varepsilon, t + \varepsilon]$. So the quasi-probability distributions $\text{qdi}S_{\mathcal{R}}$ (viewed as vectors of real numbers)⁹ are confined within a cube $[-t - \varepsilon, t + \varepsilon]^N$, where N is the number of the hidden outcomes. Since the cube is compact, from the sequence of $S_{\mathcal{R}}$ one can choose a converging subsequence, with the limit $S_{\mathcal{R}}^*$, and it is easy to see that $\|S_{\mathcal{R}}^*\|$ cannot exceed t (otherwise the original sequence of $\|S_{\mathcal{R}}\|$ would have converged to two distinct limits).

So, the set of quasi-couplings $S_{\mathcal{R}}$ for any c-c system \mathcal{R} contains a quasi-coupling $S_{\mathcal{R}}^*$ with the smallest possible value of the total variation $\|S_{\mathcal{R}}\|$. If $\|S_{\mathcal{R}}^*\|$ equals 1, then the system is noncontextual, because then $S_{\mathcal{R}}^*$ is a proper maximally connected coupling. If $\|S_{\mathcal{R}}^*\| > 1$, then no proper maximally connected coupling for \mathcal{R} exists, and the quantity $\|S_{\mathcal{R}}^*\| - 1$ can be taken as a measure (of degree) of contextuality. Note that while the minimum total variation $\|S_{\mathcal{R}}^*\|$ is unique, the quasi-coupling $S_{\mathcal{R}}^*$ generally is not.

Consider again Fig. 20. The value of $\|S_{\mathcal{R}}\|$ in it is $6 \cdot \frac{1}{2} = 3$. Is this the smallest possible value? It is not. A direct minimization of $\|S_{\mathcal{R}}\|$ subject to the linear equations $\mathbf{M}\mathbf{Q} = \mathbf{P}$ shows the minimum value in the case of the system depicted in Fig. 14 to be 2. It is reached, e.g., in the quasi-probability distribution shown in Fig. 21. This distribution therefore defines an $S_{\mathcal{R}}^*$. There are other quasi-probability distributions (in fact an infinity of them) with this minimum value of $\|S_{\mathcal{R}}\|$.

It is instructive to see how this total variation measure changes as we change the value of $p = \Pr[S_1^2 = 1, S_2^2 = 1]$ from zero to the maximal possible value $\frac{1}{2}$ while keeping all other probabilities fixed (see Fig. 22). The relationship turns out to be linear:

$$\|S_{\mathcal{R}}^*\| = 2(1 - p). \quad (72)$$

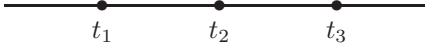
The system is maximally contextual at $p = 0$, the case we focused on in our examples. When p reaches $\frac{1}{2}$, the system is noncontextual: trivially so, because then its two bunches are identical.

A direct minimization of $\|S_{\mathcal{R}}\|$ subject to the linear equations $\mathbf{M}\mathbf{Q} = \mathbf{P}$ is a nonlinear problem, but it can be reduced to a linear programming one, in the following way:

⁹ Note that $\text{qdi}S_{\mathcal{R}}$ can be viewed as the same entity as the vector \mathbf{Q} in the matrix equation $\mathbf{M}\mathbf{Q} = \mathbf{P}$, because \mathbf{Q} is a vector of real numbers indexed by the hidden outcomes. There is a subtlety here (and throughout this paper) related to distinguishing indexed values and the pairs consisting of indexes and values, but we will ignore it.

erences, the classification of such systems into contextual and noncontextual ones, as well as into consistently connected and inconsistently connected ones, is meaningful and, at least in some applications of the theory, fundamentally important. We would like to conclude this paper by recapitulating a few points made in this paper and by offering a general observation.

(1) Inconsistency of connectedness should be distinguished from contextuality. One may call inconsistency “a kind of contextuality,” but it is contextuality of a different kind. Inconsistency of connectedness is about direct influences of certain elements of conteXts upon random variables. Such influences are revealed on the level of stochastically unrelated random variables sharing the same conteNt (i.e., within connections). Direct influences cannot act, e.g., from the future to the past or from one event to a spatially separated but simultaneous one. Contextuality, by contrast, is revealed in joint distributions of random variables, and it is not constrained by considerations of causality. As an example, consider a physical realization of the Suppes-Zanotti-Legett-Garg system (Leggett and Garg, 1985) that consists in three measurements made at three moments in time with respect to some zero point, as shown below:



The measurements are always made in pairs: at moments t_1 and t_2 or at moments t_1 and t_3 or at moments t_2 and t_3 . Each pair of times moments defines a conteXt, and each moment defines a conteNt (because the measurement in this analysis are distinguished only by the time moments at which they are made). Now, it is perfectly possible that the distribution of $R_{t_2}^{(t_1, t_2)}$ differs from the distribution of $R_{t_2}^{(t_2, t_3)}$, because in the former case the measurement at moment t_2 can be directly influenced by the fact that a measurement was made at some moment in the past, t_1 (if the system is a quantum one, its quantum state, prepared at moment zero, can be changed by a previous measurement); but a measurement cannot be influenced by another measurement yet to be made at a future moment, t_3 . By contrast, in any joint distributions of the variables, such as $(R_{t_1}^{(t_1, t_2)}, R_{t_2}^{(t_1, t_2)})$, the future random variable stochastically depends on the past one exactly whenever the past one stochastically depends on the future one. Contextuality is only revealed by looking at such joint distributions within bunches and comparing them across bunches.

(2) A distinguishing feature of Contextuality-by-Default, and the main reason for the “by-default” in its name, is that it treats random variables in different conteXts as different random variables, even if they have conteNts in common. As a result, the bunches of a system never overlap, and the problem of contextuality therefore is not posed as a problem of compatibility of different overlapping groups of random variables. Rather it is posed as a problem of compatibility between the

bunches on the one hand and maximal couplings for the connections on the other. In this respect Contextuality-by-Default is distinct from other approaches to contextuality, e.g., the prominent line of contextuality research by Abramsky and his colleagues (Abramsky & Brandenburger, 2011; Abramsky et al, 2015).

(3) Treating random variables in different conteXts as different, however, in no way means that conteXts are fused (or confused) with conteNts. On the contrary, Contextuality-by-Default is based on a strict differentiation of these entities, although, being an abstract mathematical theory, it cannot determine what constitutes conteNts and conteXts in a given empirical situation. This determination is made before the theory applies. If one changes one’s double-classification of the random variables (by the conteNts and by the conteXts), the contextuality of the system changes too.

(4) Contextuality-by-Default is not a model for empirical phenomena. As any abstract mathematical theory, it has no predictive power as a result of having no predictive intent. It is a theoretical language, on a par with, say, real analysis or probability theory. In fact, if presented in full generality to include arbitrary systems of arbitrary random variables (the presentation in this paper was confined to finite sets of categorical variables only), Contextuality-by-Default is essentially co-extensive with Kolmogorovian theory of random variables. The main difference from Kolmogorovian probability theory is that the Contextuality-by-Default theory may (but does not have to) be constructed without sample spaces, that it prominently uses the notion of stochastic unrelatedness (implicit or underemphasized in Kolmogorovian probability theory), and that the theory of couplings (rather peripheral to the mainstream Kolmogorovian theory) is at the very heart of the Contextuality-by-Default theory.

(5) In dealing with contextuality, the Contextuality-by-Default theory is about compatibility (or lack thereof) of the observed bunches of a system with maximal couplings of the separately taken connections. Maximality is not, however, the only possible constraint imposable on the couplings of the connections. Contextuality-by-Default can be expanded or modified in various ways by replacing it with other constraints, and any new constraint replacing maximality would tackle a new meaning of contextuality. Using the same logic as in Section 1.7 and in Definition 3.4, if separately taken connections can be coupled subject to some constraint C , then the system is “ C -noncontextual” if it can be coupled so that all subcouplings corresponding to its connections satisfy C ; otherwise the system is “ C -contextual.” It is remarkable that the representation of the contextuality problem as a linear programming task (Section 4) and the construction of the measure of contextuality based on the quasi-couplings (Section 6) apply with no modifications to any choice of C such that a coupling satisfying C exists for any connection taken separately. Indeed, the only property of connection probabilities required for the construction

of the matrix-vector pair $\mathbf{M}^*\mathbf{-P}^*$ (hence also $\mathbf{M-P}$) is that these probabilities exist, not the way they are computed. (The choices of \mathbf{C} for which a coupling satisfying \mathbf{C} may not exist for some connections taken separately requires a modification in the definition of contextuality, but can be handled too: any system possessing these connections can be treated as “automatically” contextual.) In choosing a constraint \mathbf{C} to replace maximality, one can be guided by certain reasonable desiderata, one of them being that \mathbf{C} should be reduced to the identity constraint when a system is consistently connected. Another reasonable desideratum could be that the “ \mathbf{C} -theory” reduces to the one described in Section 5 when specialized to cyclic systems with binary random variables. We will elaborate elsewhere.¹⁰

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- [1] Abramsky, S., Brandenburger, A. (2011). The sheaf-theoretic structure of non-locality and contextuality. *New Journal of Physics* **13**, 113036–113075.
 - [2] Abramsky, S., Barbosa, R.S., Kishida, K., Lal, R., Mansfield, S. (2015). Contextuality, cohomology and paradox. *24th EACSL Annual Conference on Computer Science Logic* (S. Kreuzer, ed.), pp. 211–228.
 - [3] [6] Al-Safi, S.W., Short, A.J. (2013). Simulating all nonsignaling correlations via classical or quantum theory with negative probabilities. *Physical Review Letters*, **111**:170403.
 - [4] Bell, J. (1964). On the Einstein-Podolsky-Rosen paradox. *Physics* **1**, 195–200.
 - [5] Bell, J. (1966). On the problem of hidden variables in quantum mechanics. *Review of Modern Physics* **38**, 447–453.
 - [6] Bohm, D, Aharonov, Y. (1957). Discussion of experimental proof for the paradox of Einstein, Rosen and Podolski. *Physical Review* **108**, 1070–1076.
 - [7] Cereceda, J. (2000). Quantum mechanical probabilities and general probabilistic constraints for Einstein–Podolsky–Rosen–Bohm experiments. *Foundations of Physics Letters* **13**, 427–442.
 - [8] Clauser, J.F., Horne, M.A., Shimony, A., Holt, R.A. (1969). Proposed experiment to test local hidden-variable theories. *Physical Review Letters* **23**, 880–884.
 - [9] Clauser, J.F., Horne, M.A. (1974). Experimental consequences of objective local theories. *Physical Review D* **10**, 526–535.
 - [10] de Barros, J.A., Oas, G. (2014). Negative probabilities and counter-factual reasoning in quantum cognition. *Physica Scripta* **T163**:014008.
 - [11] de Barros, J.A., Oas, G., Suppes, P. (2015). Negative probabilities and counterfactual reasoning on the double-slit experiment. In J.-Y. Beziau, D. Krause, J. B. Arenhart (Eds.), *Conceptual Clarification: Tributes to Patrick Suppes* (1992–2014). London: College Publications.
 - [12] de Barros, J.A., Dzhamfarov, E.N., Kujala, J.V., Oas, G. (2015). Measuring Observable Quantum Contextuality. *Lecture Notes in Computer Science* **9535**, 36–47.
 - [13] Dzhamfarov, E. N. (2003). Selective influence through conditional independence. *Psychometrika*, **68**, 7–26.
 - [14] Dzhamfarov, E.N., Kujala, J.V. (2010). The Joint Distribution Criterion and the Distance Tests for selective probabilistic causality. *Frontiers in Psychology* **1**:151 doi: 10.3389/fpsyg.2010.00151.
 - [15] Dzhamfarov, E.N., Kujala, J.V. (2012). Selectivity in probabilistic causality: Where psychology runs into quantum physics. *Journal of Mathematical Psychology* **56**, 54–63.
 - [16] Dzhamfarov, E.N., Kujala, J.V. (2014a). A qualified Kolmogorovian account of probabilistic contextuality. *Lecture Notes in Computer Science* **8369**, 201–212. 8.
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- ¹⁰ As the reviewing of this paper was nearing completing and no substantive changes could be made, we proposed a new version of CbD, with \mathbf{C} -couplings being “multimaximal” ones (Dzhamfarov and Kujala, 2016a,b): in such a coupling of a connection any of its subcouplings is a maximal coupling of the corresponding subset of the connection. If random variables in a connection are binary, their multimaximal coupling always exists and is unique. For connections with more-than-binary categorical variables, one possible approach is to replace them with all their possible dichotomizations; in each conteXt, these dichotomizations are jointly distributed and form a sub-bunch of the bunch corresponding to the context. The replacement of maximality with multimaximality affects classification of the systems into contextual and noncontextual, but it does not affect the validity of our theorems related to the measure of contextuality, as the maximality constraint was not used in their proofs. The specializations of CbD to consistently connected systems and to cyclic systems with binary random variables remain unchanged.

- [17] Dzhafarov, E.N., Kujala, J.V. (2014b). Embedding quantum into classical: contextualization vs conditionalization. *PLoS One* **9**(3): e92818. doi:10.1371/journal.pone.0092818.
- [18] Dzhafarov, E.N., Kujala, J.V. (2014c). Contextuality is about identity of random variables. *Physica Scripta* **T163**, 014009.
- [19] Dzhafarov, E.N., Kujala, J.V. (2016a). Contextuality-by-Default 2.0: Systems with binary random variables. *arXiv:1604.04799*.
- [20] Dzhafarov, E.N., Kujala, J.V. (2016a). Probabilistic foundations of contextuality. *arXiv:1604.08412*.
- [21] Dzhafarov, E.N., Kujala, J.V., Cervantes, V.H. (2016). Contextuality-by-Default: A brief overview of ideas, concepts, and terminology. *Lecture Notes in Computer Science* **9535**, 12-23.
- [22] Dzhafarov, E.N., Zhang, R., Kujala, J.V. (2015). Is there contextuality in behavioral and social systems? *Philosophical Transactions of the Royal Society A* **374**: 20150099.
- [23] Fine, A. (1982). Hidden variables, joint probability, and the Bell inequalities. *Physical Review Letters* **48**, 291–295.
- [24] Karmarkar, N. (1984). A new polynomial-time algorithm for linear programming. *Combinatorica*, **4**, 373–395.
- [25] Klyachko, A.A., Can, M.A., Binicioglu, S., Shumovsky, A.S. (2008). A simple test for hidden variables in spin-1 system. *Physical Review Letters* **101**:020403.
- [26] Kujala, J.V. (2016). Minimal distance to approximating noncontextual system as a measure of contextuality. *arXiv:1512.02340*.
- [27] Kujala, J.V., Dzhafarov, E.N. (2016). Proof of a conjecture on contextuality in cyclic systems with binary variables. *Foundations of Physics*, **46**, 282-299.
- [28] Kujala, J.V., Dzhafarov, E.N., Larsson, J.-Å. (2015). Necessary and sufficient conditions for maximal noncontextuality in a broad class of quantum mechanical systems. *Physical Review Letters* **115**:150401.
- [29] Kurzynski, P., Ramanathan, R., Kaszlikowski, D. (2012). Entropic test of quantum contextuality. *Physical Review Letters* **109**:020404.
- [30] Kurzynski, P., Cabello, A., Kaszlikowski, D. (2014). Fundamental monogamy relation between contextuality and nonlocality. *Physical Review Letters* **112**:100401.
- [31] Lapkiewicz, R., Li, P., Schaeff, C., Langford, N.K., Ramelow, S., Wieśniak, M., Zeilinger, A. (2011). Experimental non-classicality of an indivisible quantum system. *Nature* **474**, 490–493.
- [32] Leggett, A., Garg, A. (1985). Quantum mechanics versus macroscopic realism: Is the flux there when nobody looks? *Physical Review Letters* **54**:857.
- [33] Masanes, L., Acín, A., Gisin, N. (2006). General properties of nonsignaling theories. *Physical Review A* **73**: 012112.
- [34] Moore, D.W. (2002). Measuring new types of question-order effects. *Public Opinion Quarterly* **66**, 80-91.
- [35] Popescu, S., Rohrlich, D. (1994). Quantum nonlocality as an axiom. *Foundations of Physics* **24**, 379–385.
- [36] Suppes, P., Zanotti, M. (1981). When are probabilistic explanations possible? *Synthese* **48**, 191-199.
- [37] Thorisson, H. (2000). *Coupling, Stationarity, and Regeneration*. New York: Springer.
- [38] Townsend, J.T., Schweickert, R. (1989). Toward the trichotomy method of reaction times: Laying the foundation of stochastic mental networks. *Journal of Mathematical Psychology*, **33**, 309–327.
- [39] Wang, Z., Busemeyer, J.R. (2013). A quantum question order model supported by empirical tests of an a priori and precise prediction. *Topics in Cognitive Science* **5**, 689–710.
- [40] Wang, Z., Solloway T., Shiffrin, R.M., Busemeyer, J.R. (2014). Context effects produced by question orders reveal quantum nature of human judgments. *Proceedings of the National Academy of Sciences* **111**, 9431-9436.